14.

CALCULATION OF STIFFNESS AND MASS ORTHOGONAL VECTORS

LDR Vectors are Always More Accurate than Using the Exact Eigenvectors in a Mode Superposition Analysis

14.1 INTRODUCTION

The major reason to calculate mode shapes (or eigenvectors and eigenvalues) is that they are used to uncouple the dynamic equilibrium equations for mode superposition and/or response spectra analyses. The main purpose of a dynamic response analysis of a structure is ***to accurately estimate displacements and member forces*** in the real structure. In general, there is no direct relationship between the accuracy of the eigenvalues and eigenvectors and the accuracy of node point displacements and member forces.

In the early days of earthquake engineering, the Rayleigh-Ritz method of dynamic analysis was used extensively to calculate approximate solutions. With the development of high-speed computers, the use of exact eigenvectors replaced the use of Ritz vectors as the basis for seismic analysis. It will be illustrated in this book that **L**oad-**D**ependent **R**itz, LDR, vectors can be used for the dynamic analysis of both linear and nonlinear structures. The new modified Ritz method produces more accurate results, with less computational effort, than the use of exact eigenvectors.

There are several different numerical methods available for the evaluation of the eigenvalue problem. However, for large structural systems, only a few methods have proven to be both accurate and robust.

14.2 DETERMINATE SEARCH METHOD

The equilibrium equation, which governs the undamped free vibration of a typical mode, is given by:

$$[\mathbf{K} - \omega_i^2 \mathbf{M}]\mathbf{v}_i = \mathbf{0} \quad \text{or} \quad \overline{\mathbf{K}}_i \mathbf{v}_i = \mathbf{0} \tag{14.1}$$

Equation 14.1 can be solved directly for the natural frequencies of the structure by assuming values for ω_i and factoring the following equation:

$$\overline{\mathbf{K}}_i = \mathbf{L}_i \mathbf{D}_i \mathbf{L}_i^T \tag{14.2}$$

From Appendix C the determinant of the factored matrix is defined by:

$$\text{Det}(\omega_i) = D_{11} D_{22} \text{----} D_{NN} \tag{14.3}$$

It is possible, by repeated factorization, to develop a plot of the determinant vs. λ , as shown in Figure 14.1. This classical method for evaluating the natural frequencies of a structure is called the ***determinant search method*** [1]. It should be noted that for matrices with small bandwidths the numerical effort to factor the matrices is very small. For this class of problem the determinant search method, along with inverse iteration, is an effective method of evaluating the undamped frequencies and mode shapes for small structural systems. However, because of the increase in computer speeds, small problems can be solved by any method in a few seconds. Therefore, the determinant search method is no longer used in modern dynamic analysis programs.

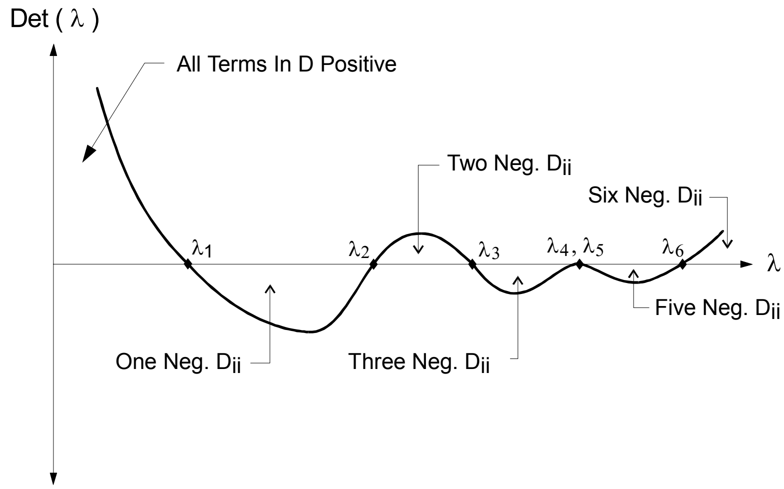

Figure 14.1 Determinant vs. Frequency for Typical System

14.3 STURM SEQUENCE CHECK

Figure 14.1 illustrates a very important property of the sequence of diagonal terms of the factored matrix. One notes that for a specified value of ω_i , one can count the number of negative terms in the diagonal matrix and it is always equal to the number of frequencies below that value. Therefore, it can be used to check a method of solution that fails to calculate all frequencies below a specified value. Also, another important application of the Sturm Sequence Technique is to evaluate the number of frequencies within a frequency range. It is only necessary to factor the matrix at both the maximum and minimum frequency points, and the difference in the number of negative diagonal terms is equal to the number of frequencies in the range. This numerical technique is useful in machine vibration problems.

14.4 INVERSE ITERATION

Equation (14.1) can be written in an iterative solution form as:

$$K\overline{V}_n^{(i)} = \lambda_n^{(i-1)} M V_n^{(i-1)} \quad \text{or} \quad LDL^T \overline{V}_n^{(i)} = R^{(i)} \tag{14.4}$$

The computational steps required for the solution of one eigenvalue and eigenvector can be summarized as follows:

1. Factor stiffness matrix into triangularized $\mathbf{LDL}^T$ form during static load solution phase.

2. For the first iteration, assume $\mathbf{R}^{(1)}$ to be a vector of random numbers and solve for initial vector $\overline{\mathbf{V}}_n^{(1)}$.

3. Iterate with i = 1, 2 . . .

   a. Normalize vector so that $\mathbf{V}_n^{T(i)}\mathbf{M}\mathbf{V}_n^{(i)} = \mathbf{I}$

   b. Estimate eigenvalue $\lambda_n^{(i)} = \mathbf{V}_n^{T(i)}\mathbf{R}^{(i)}$

   c. Check $\lambda_n^{(i)}$ for convergence - if converged, terminate

   d. i = i + 1 and calculate $\mathbf{R}^{(i)} = \lambda_n^{(i-1)}\mathbf{M}\mathbf{V}^{(i-1)}$

   e. Solve for new vector $\mathbf{LDL}^T\overline{\mathbf{V}}_n^{(i)} = \mathbf{R}^{(i)}$

   f. Repeat Step 3

It can easily be shown that this method will converge to the smallest unique eigenvalue.

14.5 GRAM-SCHMIDT ORTHOGONALIZATION

Additional eigenvectors can be calculated using the inverse iteration method if, after each iteration cycle, the iteration vector is made orthogonal to all previously calculated vectors. To illustrate the method, let us assume that we have an approximate vector $\overline{\mathbf{V}}$ that needs to be made orthogonal to the previously calculated vector $\mathbf{V}_n$. Or, the new vector can be calculated from:

$$\mathbf{V} = \overline{\mathbf{V}} - \alpha\mathbf{V}_n \tag{14.5}$$

Multiplying Equation (14.3) by $\mathbf{V}_n^T\mathbf{M}$, we obtain:

$$V_n^T M V = V_n^T M \overline{V} - \alpha V_n^T M V_n = 0 \tag{14.6}$$

Therefore, the orthogonality requirement is satisfied if:

$$\alpha = \frac{V_n^T M \overline{V}}{V_n^T M V_n} = V_n^T M \overline{V} \tag{14.7}$$

If the orthogonalization step is inserted after Step 3.e in the inverse iteration method, additional eigenvalues and vectors can be calculated.

14.6 BLOCK SUBSPACE ITERATION

Inverse iteration with one vector may not converge if eigenvalues are identical and the eigenvectors are not unique. This case exists for many real three-dimensional structures, such as buildings with equal stiffness and mass in the principle directions. This problem can be avoided by iterating with a block of orthogonal vectors [2]. The block subspace iteration algorithm is summarized in Table 14.1 and is the method used in the modern versions of the SAP program.

Experience has indicated that the subspace block size "b" should be set equal to the square root of the *average bandwidth of the stiffness matrix*, but, not less than six. The block subspace iteration algorithm is relatively slow; however, it is very accurate and robust. In general, after a vector is added to a block, it requires five to ten forward reductions and back-substitutions before the iteration vector converges to the exact eigenvector.

Table 14.1 Subspace Algorithm for the Generation of Eigenvectors

I. INITIAL CALCULATIONS

A. Triangularize Stiffness Matrix.

B. Use random numbers to form a block of “b” vectors $V^{(0)}$.

II. GENERATE L EIGENVECTORS BY ITERATION i = 1,2...

A. Solve for block of vectors, $X^{(i)}$ in, $K X^{(i)} = M V^{(i-1)}$.

B. Make block of vectors, $X^{(i)}$, stiffness and mass orthogonal, $\overline{V}^{(i)}$. Order eigenvalues and corresponding vectors in ascending order.

C. Use Gram-Schmidt method to make $\overline{V}^{(i)}$ orthogonal to all previously calculated vectors and normalized so that $V^{T(i)} M V^{(i)} = I$.

D. Perform the following checks and operations:

1. If first vector in block is not converged, go to Step A with **i = i + 1** .
2. Save Vector ϕ_n on Disk.
3. If n equals L , terminate iteration.
4. Compact block of vectors.
5. Add random number vector to last column of block.

Return to Step D.1 with **n = n + 1**

14.7 SOLUTION OF SINGULAR SYSTEMS

For a few types of structures, such as aerospace vehicles, it is not possible to use inverse or subspace iteration directly to solve for mode shapes and frequencies. This is because there is a minimum of six rigid-body modes with zero frequencies and the stiffness matrix is singular and cannot be triangularized. To solve this problem, it is only necessary to introduce the following eigenvalue ***shift,*** or change of variable:

$$\lambda_n = \overline{\lambda}_n - \rho \tag{14.8}$$

Hence, the iterative eigenvalue problem can be written as:

$$\overline{\mathbf{K}}\,\overline{\mathbf{V}}_n^{(i)} = \overline{\lambda}_n^{(i-1)}\mathbf{M}\mathbf{V}_n^{(i-1)} \quad \text{or} \quad \mathbf{LDL}^T\overline{\mathbf{V}}_n^{(i)} = \mathbf{R}^{(i)} \tag{14.9}$$

The shifted stiffness matrix is now non-singular and is defined by:

$$\overline{\mathbf{K}} = \mathbf{K} + \rho\mathbf{M} \tag{14.10}$$

The eigenvectors are not modified by the arbitrary shift ρ. The correct eigenvalues are calculated from Equation (14.8).

14.8 GENERATION OF LOAD-DEPENDENT RITZ VECTORS

The numerical effort required to calculate the exact eigen solution can be enormous for a structural system if a large number of modes are required. However, many engineers believe that this computational effort is justifiable if accurate results are to be obtained. One of the purposes of this section is to clearly illustrate that this assumption is not true for the dynamic response analyses of all structural systems.

It is possible to use the exact free-vibration mode shapes to reduce the size of both linear and nonlinear problems. However, this is not the best approach for the following reasons:

1. For large structural systems, the solution of the eigenvalue problem for the free-vibration mode shapes and frequencies can require a significant amount of computational effort.

2. In the calculation of the free-vibration mode shapes, the spatial distribution of the loading is completely disregarded. Therefore, many of the mode shapes that are calculated are orthogonal to the loading and do not participate in the dynamic response.

3. If dynamic loads are applied at massless degrees-of-freedom, the use of all the exact mode shapes in a mode superposition analysis will not converge to the exact solution. In addition, displacements and stresses near the application of the loads can be in significant error. Therefore, there is no need to apply the "static correction method" as would be required if exact eigenvectors are used for such problems.

4. It is possible to calculate a set of stiffness and mass orthogonal Ritz vectors, with a minimum of computational effort, which will converge to the exact solution for any spatial distribution of loading [2].

It can be demonstrated that a dynamic analysis based on a unique set of Load Dependent Vectors yields a more accurate result than the use of the same number of exact mode shapes. The efficiency of this technique has been illustrated by solving many problems in structural response and in wave propagation types of problems [4]. Several different algorithms for the generation of Load Dependent Ritz Vectors have been published since the method was first introduced in 1982 [3]. Therefore, it is necessary to present in Table 14.2 the latest version of the method for multiple load conditions.

Table 14.2 Algorithm for Generation of Load Dependent Ritz Vectors

I. INITIAL CALCULATIONS

A. Triangularize Stiffness Matrix $\mathbf{K} = \mathbf{L}^T \mathbf{D} \mathbf{L}$.

B. Solve for block of "b" static displacement vectors u_s resulting from spatial load patterns $\mathbf{F}$; or, $\mathbf{K}\,\mathbf{u}_s = \mathbf{F}$.

C. Make block of vectors $\mathbf{u}_s$, stiffness and mass orthogonal, $\mathbf{V}_1$.

II. GENERATE BLOCKS OF RITZ VECTORS i = 2,....N

A. Solve for block of vectors, $\mathbf{X}_i$, $\mathbf{K}\,\mathbf{X}_i = \mathbf{M}\,\mathbf{V}_{i-1}$.

B. Make block of vectors, $\mathbf{X}_i$, stiffness and mass orthogonal, $\overline{\mathbf{V}}_i$.

Table 14.2 Algorithm for Generation of Load Dependent Ritz Vectors

C. Use Modified Gram-Schmidt method (two times) to make $\overline{\mathbf{V}}_i$ orthogonal to all previously calculated vectors and normalized so that $\mathbf{V}_i^T \mathbf{M} \mathbf{V}_i = \mathbf{I}$.

III. MAKE VECTORS STIFFNESS ORTHOGONAL

A. Solve Nb by Nb eigenvalue problem $[\overline{\mathbf{K}} - \Omega^2 \mathbf{I}]\mathbf{Z} = 0$ where $\overline{\mathbf{K}} = \mathbf{V}^T \mathbf{K} \mathbf{V}$.

B. Calculate stiffness orthogonal Ritz vectors, $\Phi = \mathbf{V}\mathbf{Z}$.

14.9 A PHYSICAL EXPLANATION OF THE LDR ALGORITHM

The physical foundation for the method is the recognition that the dynamic response of a structure will be a function of the spatial load distribution. The undamped, dynamic equilibrium equations of an elastic structure can be written in the following form:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{R}(t) \tag{14.11}$$

In the case of earthquake or wind, the time-dependent loading acting on the structure, $\mathbf{R}(t)$, Equation (13.1), can be written as:

$$\mathbf{R}(t) = \sum_{j=1}^{J} \mathbf{f}_j \mathbf{g}(t)_j = \mathbf{F}\mathbf{G}(t) \tag{14.12}$$

Note that the independent load patterns $\mathbf{F}$ are not a function of time. For constant earthquake ground motions at the base of the structure three independent load patterns are possible. These load patterns are a function of the directional mass distribution of the structure. In case of wind loading, the downwind mean wind pressure is one of those vectors. The time functions $\mathbf{G}(t)$ can always be expanded into a Fourier series of sine and cosine functions. Hence, neglecting damping, a typical dynamic equilibrium equation to be solved is of the following form:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}\sin\overline{\varpi}\,t \tag{14.13}$$

Therefore, the exact dynamic response for a typical loading frequency ϖ is of the following form:

$$\mathbf{K}\mathbf{u} = \mathbf{F} + \varpi^2\,\mathbf{M}\mathbf{u} \tag{14.14}$$

This equation cannot be solved directly because of the unknown frequency of the loading. However, a series of stiffness and mass orthogonal vectors can be calculated that will satisfy this equation using a perturbation algorithm. The first block of vectors is calculated by neglecting the mass and solving for the static response of the structure. Or:

$$\mathbf{K}\mathbf{u}_0 = \mathbf{F} \tag{14.15}$$

From Equation (14.14) it is apparent that the distribution of the error in the solution, due to neglecting the inertia forces, can be approximated by:

$$\mathbf{F}_1 \approx \mathbf{M}\mathbf{u}_\mathbf{0} \tag{14.16}$$

Therefore, an additional block of displacement error, or correction, vectors can be calculated from:

$$\mathbf{K}\mathbf{u}_1 = \mathbf{F}_1 \tag{14.17}$$

In calculating $\mathbf{u}_1$ the additional inertia forces are neglected. Hence, in continuing this thought process, it is apparent the following recurrence equation exists:

$$\mathbf{K}\mathbf{u}_i = \mathbf{M}\mathbf{u}_{i-1} \tag{14.18}$$

A large number of blocks of vectors can be generated by Equation (14.18). However, to avoid numerical problems, the vectors must be stiffness and mass orthogonal after each step. In addition, care should be taken to make sure that all vectors are linearly independent. The complete numerical algorithm is summarized in Table 14.2. After careful examination of the LDR vectors, one can conclude that ***dynamic analysis is a simple extension of static analysis*** because the first block of vectors is the static response from all load patterns acting on the structure. For the case where loads are applied at only the mass degrees-of-freedom, the LDR vectors are always a linear combination of the exact eigenvectors.

It is of interest to note that the recursive equation, used to generate the LDR vectors, is similar to the Lanczos algorithm for calculating exact eigenvalues and vectors, except that the starting vectors are the static displacements caused by the spatial load distributions. Also, *there is no iteration involved in the generation of Load Dependent Ritz vectors*.

14.10 COMPARISON OF SOLUTIONS USING EIGEN AND RITZ VECTORS

The fixed-end beam shown in Figure 14.1 is subjected to a point load at the center of the beam. The load varies in time as a constant unit step function.

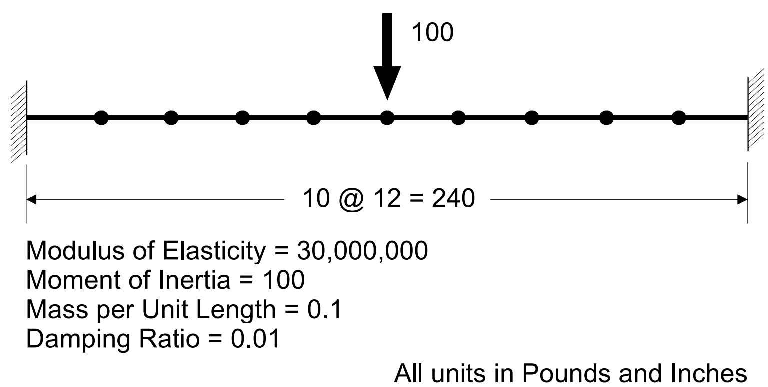

Figure 14.1 Dimensions, Stiffness and Mass for Beam Structure

The damping ratio for each mode was set at one percent and the maximum displacement and moment occur at 0.046 second, as shown in Table 14.3.

The results clearly indicate the advantages of using load-dependent vectors. One notes that the free-vibration modes 2, 4, 6 and 8 are not excited by the loading because they are nonsymmetrical. However, the load dependent algorithm generates only the symmetrical modes. In fact, the algorithm will fail for this case, if more than five vectors are requested.

Table 14.3 Results from Dynamic Analyses of Beam Structure

Number of Vectors	Free-Vibration Mode Shapes		Load-Dependent Ritz Vectors	
	Displacement	Moment	Displacement	Moment
1	0.004572 (-2.41)	4178 (-22.8)	0.004726 (+0.88)	5907 (+9.2)
2	0.004572 (-2.41)	4178 (-22.8)	0.004591 (-2.00)	5563 (+2.8)
3	0.004664 (-0.46)	4946 (-8.5)	0.004689 (+0.08)	5603 (+3.5)
4	0.004664 (-0.46)	4946 (-8.5)	0.004688 (+0.06)	5507 (+1.8)
5	0.004681 (-0.08)	5188 (-4.1)	0.004685 (0.00)	5411 (0.0)
7	0.004683 (-0.04)	5304 (-2.0)		
9	0.004685 (0.00)	5411 (0.0)		

Note: Numbers is parentheses are percentage errors.

Both methods give good results for the maximum displacement. The results for maximum moment, however, indicate that the load-dependent vectors give significantly better results and converge from above the exact solution. It is clear that free-vibration mode shapes are not necessarily the best vectors to be used in mode-superposition dynamic response analysis. Not only is the calculation of the exact free-vibration mode shapes computationally expensive, it requires more vectors, which increases the number of modal equations to be integrated and stored within the computer.

14.11 CORRECTION FOR HIGHER MODE TRUNCATION

In the analysis of many types of structures, the response of higher modes can be significant. In the use of exact eigenvectors for mode superposition or response spectra analyses, approximate methods of analysis have been developed to improve the results. The purpose of those approximate methods is “to account for missing mass” or “to add static response” associated with “higher mode truncation.” Those methods are used to reduce the number of exact eigenvectors to be calculated, which reduces computation time and computer storage requirements.

The use of Load Dependent Ritz, LDR, vectors, on the other hand, does not require the use of those approximate methods because the “static response” is included in the initial set of vectors. This is illustrated by the time history analysis of a simple cantilever structure subjected to earthquake motions shown in Figure 14.2. This is a model of a light-weight superstructure built on a massive foundation supported on stiff piles that are modeled using a spring.

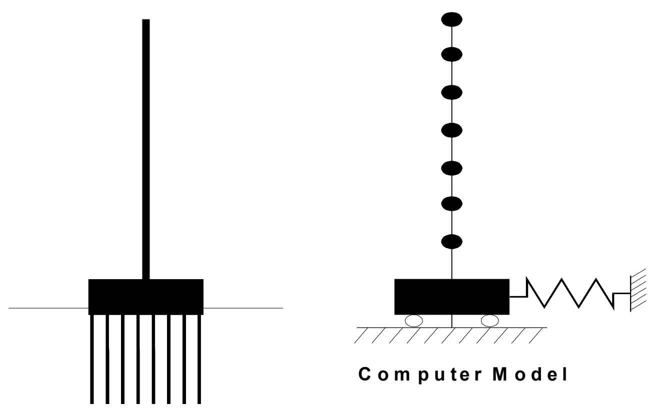

Figure 14.2 Cantilever Structure on Massive Stiff Foundation

Only eight eigen or Ritz vectors can be used because the model has only eight masses. The computed periods, using the exact eigen or Ritz method, are summarized in Table 14.4. It is apparent that the eighth mode is associated with the vibration of the foundation mass and the period is very short: 0.00517 seconds.

Table 14.4 Periods and Mass Participation Factors

MODE NUMBER	PERIOD (Seconds)	MASS PARTICIPATION (Percentage)
1	1.27321	11.706
2	0.43128	01.660
3	0.24205	00.613
4	0.16018	00.310
5	0.11899	00.208
6	0.09506	00.100
7	0.07951	00.046
8	0.00517	85.375

The maximum foundation force using different numbers of eigen and LDR vectors is summarized in Table 14.5. In addition, the total mass participation associated with each analysis is shown. The integration time step is the same as the earthquake motion input; therefore, no errors are introduced other than those resulting from mode truncation. Five percent damping is used in all cases.

Table 14.5 Foundation Forces and Total Mass Participation

NUMBER OF VECTORS	FOUNDATION FORCE (Kips)		MASS PARTICIPATION (Total Percentage)	
	EIGEN	RITZ	EIGEN	RITZ
8	1,635	1,635	100.0	100.0
7	260	1,636	14.6	83.3
5	259	1,671	14.5	16.2
3	258	1,756	14.0	14.5
2	257	3,188	13.4	13.9

The solution for eight eigen or LDR vectors produces the exact solution for the foundation force and 100 percent of the participating mass. For seven eigenvectors, the solution for the foundation force is only 16 percent of the exact value—a significant error; whereas, the LDR solution is almost identical to the exact foundation force. It is of interest to note that the LDR method overestimates the force as the number of vectors is reduced—a conservative engineering result.

Also, it is apparent that the mass participation factors associated with the LDR solutions are not an accurate estimate the error in the foundation force. In this case, 90 percent mass participation is not a requirement if LDR vectors are used. If only five LDR vectors are used, the total mass participation factor is only 16.2 percent; however, the foundation force is over-estimated by 2.2 percent.

14.12 VERTICAL DIRECTION SEISMIC RESPONSE

Structural engineers are required for certain types of structures, to calculate the vertical dynamic response. During the past several years, many engineers have told me that it was necessary to calculate several hundred mode shapes for a large structure to obtain the 90 percent mass participation in the vertical direction. In all cases, the "exact" free vibration frequencies and mode shapes were used in the analysis.

To illustrate this problem and to propose a solution, a vertical dynamic analysis is conducted of the two dimensional frame shown in Figure 14.3. The mass is lumped at the 35 locations shown; therefore, the system has 70 possible mode shapes.

Using the exact eigenvalue solution for frequencies and mode shapes, the mass participation percentages are summarized in Table 14.6.

One notes that the lateral and vertical modes are uncoupled for this very simple structure. Only two of the first ten modes are in the vertical direction. Hence, the total vertical mass participation is only 63.3 percent.

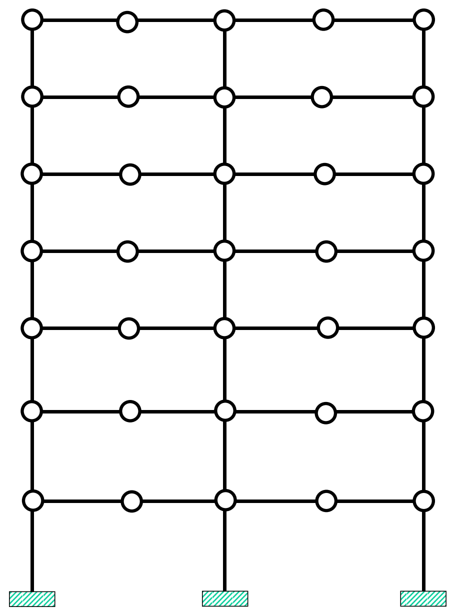

Figure 14.3 Frame Structure Subjected to Vertical Earthquake Motions

Table 14.6 Mass Participation Percentage Factors for Exact Eigenvalues

MODE	PERIOD (Seconds)	LATERAL MASS PARTICIPATION		VERTICAL MASS PARTICIPATION	
		EACH MODE	TOTAL	EACH MODE	TOTAL
1	1.273	79.957	79.957	0	0
2	0.421	11.336	91.295	0	0
3	0.242	4.172	95.467	0	0
4	0.162	1.436	96.903	0	0
5	0.158	0.650	97.554	0	0
6	0.148	0	97.554	60.551	60.551
7	0.141	0.031	97.584	0	60.551
8	0.137	0.015	97.584	0	60.551
9	0.129	0.037	97.639	0	60.551
10	0.127	0	97.639	2.775	63.326

The first 10 **L**oad **D**ependent **R**itz vectors are calculated and the mass participation percentages are summarized in Table 14.7. The two starting LDR vectors were generated using static loading proportional to the lateral and vertical mass distributions.

Table 14.7 Mass Participation Percentage Factors Using LDR Vectors

MODE	PERIOD (Seconds)	LATERAL MASS PARTICIPATION		VERTICAL MASS PARTICIPATION	
		EACH MODE	TOTAL	EACH MODE	TOTAL
1	1.273	79.957	79.957	0	0
2	0.421	11.336	91.295	0	0
3	0.242	4.176	95.471	0	0
4	0.158	2.388	97.859	0	0
5	0.149	0	97.859	60.567	60.567
6	0.123	0	97.859	4.971	65.538
7	0.104	2.102	99.961	0	65.538
8	0.103	0	99.961	13.243	78.781
9	0.064	0	99.961	9.696	88.477
10	0.041	0	99.961	8.463	96.940

The ten vectors produced by the LDR method more than satisfy the 90 percent code requirement. It would require the calculation of 34 eigenvectors for the exact eigenvalue approach to obtain the same mass participation percentage. This is just one additional example of why use of the LDR method is superior to the use of the exact eigenvectors for seismic loading.

The reason for the impressive accuracy of the LDR method compared to the exact eigenvector method is that only the mode shapes that are excited by the seismic loading are calculated.

14.13 SUMMARY

There are three different mathematical methods for the numerical solution of the eigenvalue problem. They all have advantages for certain types of problems.

First, the determinant search method, which is related to finding the roots of a polynomial, is a fundamental traditional method. It is not efficient for large structural problems. The Sturm sequence property of the diagonal elements of the factored matrix can be used to determine the number of frequencies of vibration within a specified range.

Second, the inverse and subspace iteration methods are subsets of a large number of power methods. The Stodola method is a power method. However, the use of a *sweeping matrix* to obtain higher modes is not practical because it eliminates the sparseness of the matrices. Gram-Schmidt orthogonalization is the most effective method to force iteration vectors to converge to higher modes.

Third, transformation methods are very effective for the calculation of all eigenvalues and eigenvectors of small dense matrices. Jacobi, Givens, Householder, Wilkinson and Rutishauser are all well-known transformation methods. The author prefers to use a modern version of the Jacobi method in the ETABS and SAP programs. It is not the fastest; however, we have found it to be accurate and robust. Because it is only used for problems equal to the size of the subspace, the computational time for this phase of the solution is very small compared to the time required to form the subspace eigenvalue problem. The derivation of the Jacobi method is given in Appendix D.

The use of Load Dependent Ritz vectors is the most efficient approach to solve for accurate node displacements and member forces within structures subjected to dynamic loads. The lower frequencies obtained from a Ritz vector analysis are always very close to the exact free vibration frequencies. If frequencies and mode shapes are missed, it is because the dynamic loading does not excite them; therefore, they are of no practical value. Another major advantage of using LDR vectors is that it is not necessary to be concerned about errors introduced by higher mode truncation of a set of exact eigenvectors.

All LDR mode shapes are linear combinations of the exact eigenvectors; therefore, the method always converges to the exact solution. Also, the computational time required to calculate the LDR vectors is significantly less than the time required to solve for eigenvectors.

14.14 REFERENCES

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