

5.

ISOPARAMETRIC ELEMENTS

Bruce Irons, in 1968, Revolutionized the Finite Element Method by Introducing a Natural Coordinate Reference System

5.1 INTRODUCTION

Before development of the Finite Element Method, researchers in the field of structural engineering and structural mechanics found “closed form” solutions in terms of known mathematical functions of many problems in continuum mechanics. However, practical structures of arbitrary geometry, non-homogeneous materials or structures made of several different materials are difficult to solve by this classical approach.

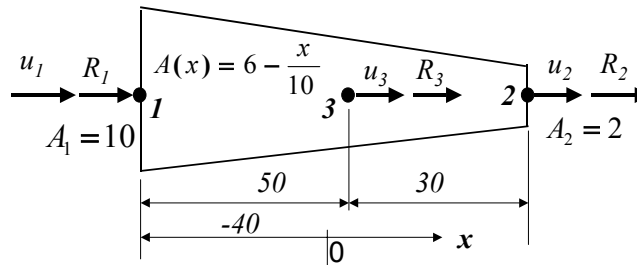
Professor Ray Clough coined the terminology “Finite Element Method” in a paper presented in 1960 [1]. This paper proposed to use the method as an alternative to the finite difference method for the numerical solution of stress concentration problems in continuum mechanics. The major purpose of the earlier work at the Boeing Airplane Company published in 1956 [2] was to include the skin stiffness in the analysis of wing structures and was not intended to accurately calculate stresses in continuous structures. The first, fully automated, finite element computer program was developed during the period of 1961 - 1962 [3].

It is the author’s opinion that the introduction of the isoparametric element formulation in 1968 by Bruce Irons [4] was the single most significant

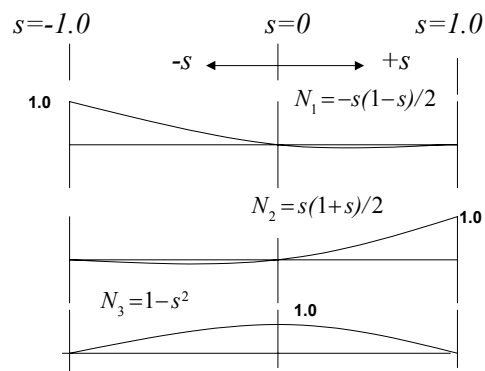
contribution to the field of finite element analysis during the past 40 years. It allowed very accurate, higher-order elements of arbitrary shape to be developed and programmed with a minimum of effort. The addition of incompatible displacement modes to isoparametric elements in 1971 was an important, but minor, extension to the formulation [5].

5.2 A SIMPLE ONE-DIMENSIONAL EXAMPLE

To illustrate the fundamentals of the isoparametric approach, the one-dimensional, three-node element shown in Figure 5.1 is formulated in a natural coordinate reference system.



A. GLOBAL REFERENCE SYSTEM "x"



B. ISOPARAMETRIC REFERENCE SYSTEM "s"

Figure 5.1 A Simple Example of an Isoparametric Element

The shape functions N_i are written in terms of the element isoparametric reference system. The "natural" coordinate s has a range of $s = \pm 1.0$. The isoparametric and global reference systems are related by the following elementary equation:

$$x(s) \equiv N_1(s) x_1 + N_2(s) x_2 + N_3(s) x_3 = \mathbf{N}(s) \mathbf{x} \quad (5.1)$$

The validity of this equation can be verified at values of $s = -1$, $s = 0$ and $s = 1$. No additional mathematical references are required to understand Equation (5.1).

The global displacement can now be expressed in terms of the fundamental isoparametric shape functions. Or:

$$u(s) = N_1(s) u_1 + N_2(s) u_2 + N_3(s) u_3 = \mathbf{N}(s) \mathbf{u} \quad (5.2)$$

Note that the sum of the shape functions is equal to 1.0 for all values of s ; therefore, rigid-body displacement of the element is possible. This is a fundamental requirement of all displacement approximations for all types of finite elements.

The strain-displacement equation for this one-dimensional element is:

$$\varepsilon_x = \frac{\partial u(s)}{\partial x} = \frac{du(s)}{dx} = \frac{du(s)}{ds} \frac{ds}{dx} \quad (5.3)$$

You may recall from sophomore calculus that this is a form of the *chain rule*. For any value of s the following equations can be written:

$$\frac{du(s)}{ds} = \mathbf{N}(\mathbf{s}),_s \mathbf{u} \quad (5.4a)$$

$$\frac{dx}{ds} = \mathbf{N}(\mathbf{s}),_s \mathbf{x} = J(s) \quad (5.4b)$$

Therefore:

$$\varepsilon_x = \frac{du(s)}{ds} \frac{ds}{dx} = \frac{1}{J(s)} \mathbf{N}(s)_{,s} \mathbf{u} = \mathbf{B}(s) \mathbf{u} \quad (5.5)$$

From Equation (5.1), the derivatives with respect to the global and isoparametric reference systems are related by:

$$dx = \mathbf{N}(s)_{,s} \mathbf{x} ds = J(s) ds \quad (5.6)$$

The 3 by 3 element stiffness can now be expressed in terms of the natural system:

$$\mathbf{K} = \int_{-1}^{+1} \mathbf{B}(s)^T \mathbf{E} \mathbf{B}(s) J(s) ds \quad (5.7)$$

In general, Equation (5.7) cannot be evaluated in closed form. However, it can be accurately evaluated by numerical integration.

5.3 ONE-DIMENSIONAL INTEGRATION FORMULAS

Most engineers have used Simpson's rule or the trapezoidal rule to integrate a function evaluated at equal intervals. However, those traditional methods are not as accurate, for the same computational effort, as the Gauss numerical integration method presented in Appendix G. The Gauss integration formulas are of the following form:

$$I = \int_{-1}^{+1} f(s) ds = \sum_{i=1}^n W_i f(s_i) \quad (5.8)$$

The Gauss points and weight factors for three different formulas are summarized in Table 5.1.

Table 5.1 Gauss Points and Weight Factors for Numerical Integration

n	s_1	W_1	s_2	W_2	s_3	W_3
1	0	2				

2	$-1/\sqrt{3}$	1	$1/\sqrt{3}$	1		
3	$-\sqrt{0.6}$	5/9	0	8/9	$\sqrt{0.6}$	5/9

Note that the sum of the weight factors is always equal to 2. Higher order numerical integration formulas are possible. However, for most displacement-based finite element analysis higher order integration is not required. In fact, for many elements, lower order integration produces more accurate results than higher order integration.

For the analysis of the tapered beam, shown in Figure 5.1, the same material properties, loading and boundary conditions are used as were used for the example presented in Section 4.2. The results are summarized in Table 5.2.

Table 5.2 Summary of Results of Tapered Rod Analyses

ELEMENT TYPE	Integration Order	u_3 (%error)	σ_1 (%error)	σ_2 (%error)	σ_3 (%error)
EXACT		0.1607	1.00	5.00	2.00
Constant Strain	Exact	0.1333 (-17.1 %)	1.67 (+67 %)	1.67 (-66 %)	1.67 (-16.5 %)
3-node isoparametric	2 point	0.1615 (+0.5 %)	0.58 (-42 %)	4.04 (-19 %)	2.31 (+15.5 %)
3-node isoparametric	3 point	0.1609 (+0.12 %)	0.83 (-17 %)	4.67 (-6.7 %)	2.76 (+34 %)

From this simple example, the following conclusions and remarks can be made:

1. Small errors in displacement do not indicate small errors in stresses.
2. Lower order integration produces a more flexible structure than the use of higher order numerical integration.
3. If this isoparametric element is integrated exactly, the tip displacement would be less than the exact displacement.

4. The stresses were calculated at the integration point and extrapolated to the nodes. Every computer program uses a different method to evaluate the stresses within an element. Those methods will be discussed later.

5.4 RESTRICTION ON LOCATIONS OF MID-SIDE NODES

The previous example illustrates that the location of the mid-side node need not be at the geometric center of the element. However, its location is not completely arbitrary.

Equation (5.4b) can be rewritten, with $x_1 = -\frac{L}{2}$, $x_2 = \frac{L}{2}$ and $x_3 = r\frac{L}{2}$, as

$$J(s) = (2 - sr)\frac{L}{2} \quad (5.9)$$

where r is the relative location of node 3, with respect to the center of the element. Equation (5.5) indicates that the strains can be infinite if $J(s)$ is zero. Also, if $J(s)$ is negative, it implies that the coordinate transformation between x and s is very distorted. For infinite strains at locations $s = \pm 1$, the zero singularity can be found from:

$$2 \pm r = 0, \text{ or } r = \pm \frac{1}{2} \quad (5.10)$$

Hence, the mid-side node location must be within the middle one-half of the element. In the case of two- and three-dimensional elements, ***mid-side nodes should be located within the middle one-half of each edge or side.***

At a crack tip, where the physical strains can be very large, it has been proposed that the elements adjacent to the crack have the mid-side node located at one-fourth the length of the element side. The stresses at the integration points will then be realistic; and element strain energy can be estimated, which may be used to predict crack propagation or stability [5].

5.5 TWO-DIMENSIONAL SHAPE FUNCTIONS

Two-dimensional shape functions can be written for different elements with an arbitrary number of nodes. The formulation presented here will be for a general four-sided element with four to nine nodes. Therefore, one formulation will cover all element types shown in Figure 5.2.

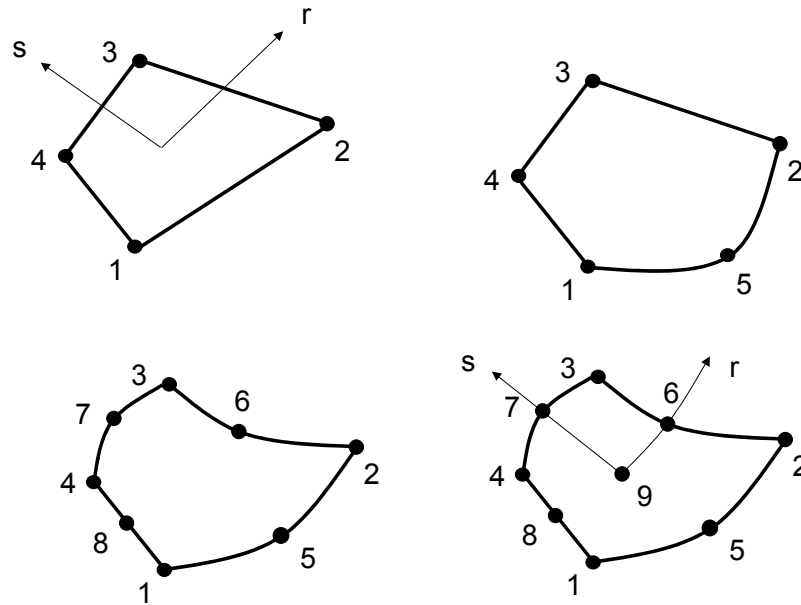


Figure 5.2 Four- to Nine-Node Two-Dimensional Isoparametric Elements

The shape functions, in the natural r - s system, are a product of the one-dimensional functions shown in Figure 5.1. The range of both r and s is ± 1 . All functions must equal 1.0 at the node and equal zero at all other nodes associated with the element. The shape functions shown in Table 5.3 are for the basic four-node element. The table indicates how the functions are modified if nodes 5, 6, 7, 8 or 9 exist.

Table 5.3 Shape Functions for a Four- to Nine-Node 2D Element

NODE i	r_i	s_i	SHAPE FUNCTION $N_1(r, s)$	OPTIONAL NODES				
				5	6	7	8	9
1	-1	-1	$N_1 = (1-r)(1-s)/4$	$-\frac{N_5}{2}$			$-\frac{N_8}{2}$	$-\frac{N_9}{4}$
2	1	-1	$N_2 = (1+r)(1-s)/4$	$-\frac{N_5}{2}$	$-\frac{N_6}{2}$			$-\frac{N_9}{4}$
3	1	1	$N_3 = (1+r)(1+s)/4$		$-\frac{N_6}{2}$	$-\frac{N_7}{2}$		$-\frac{N_9}{4}$
4	-1	1	$N_4 = (1-r)(1+s)/4$			$-\frac{N_7}{2}$	$-\frac{N_8}{2}$	$-\frac{N_9}{4}$
5	0	-1	$N_5 = (1-r^2)(1-s)/2$					$-\frac{N_9}{2}$
6	1	0	$N_6 = (1+r)(1-s^2)/2$					$-\frac{N_9}{2}$
7	0	1	$N_7 = (1-r^2)(1+s)/2$					$-\frac{N_9}{2}$
8	-1	0	$N_8 = (1-r)(1-s^2)/2$					$-\frac{N_9}{2}$
9	0	0	$N_9 = (1-r^2)(1-s^2)$					

If any node from 5 to 9 does not exist, the functions associated with that node are zero and need not be calculated. Note the sum of all shape functions is always equal to 1.0 for all points within the element. Tables with the same format can be created for the derivatives of the shape functions $N_{i,r}$ and $N_{i,s}$. The shape functions and their derivatives are numerically evaluated at the integration points.

The relationship between the natural r - s and local orthogonal x - y systems are by definition:

$$x(r, s) = \sum N_i x_i \quad (5.11a)$$

$$y(r, s) = \sum N_i y_i \quad (5.11b)$$

Also, the local x and y displacements are assumed to be of the following form:

$$u_x(r, s) = \sum N_i u_{xi} \quad (5.12a)$$

$$u_y(r, s) = \sum N_i u_{yi} \quad (5.12b)$$

To calculate strains it is necessary to take the derivatives of the displacements with respect to x and y . Therefore, it is necessary to use the classical chain rule, which can be written as:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} \\ \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \end{aligned} \quad \text{or} \quad \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} \quad (5.13)$$

The matrix \mathbf{J} is known in mathematics as the **Jacobian matrix** and can be numerically evaluated from:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \begin{bmatrix} \sum N_{i,r} x_i & \sum N_{i,r} y_i \\ \sum N_{i,s} x_i & \sum N_{i,s} y_i \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \quad (5.14)$$

At the integration points the \mathbf{J} matrix can be numerically inverted. Or:

$$\mathbf{J}^{-1} = \frac{1}{J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \quad (5.15)$$

The term J is the determinate of the Jacobian matrix and is:

$$J = J_{11} J_{22} - J_{12} J_{21} = \frac{\partial x}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial x}{\partial s} \frac{\partial y}{\partial r} \quad (5.16)$$

Figure 5.3 illustrates the physical significance of this term at any point r and s within the element. Simple geometry calculations will illustrate that J relates the area in the x - y system to the natural reference system. Or:

$$dA = dx \, dy = J \, dr \, ds \quad (5.17)$$

Hence, all the basic finite element equations can be transformed into the natural reference system and standard numerical integration formulas can be used to evaluate the integrals.

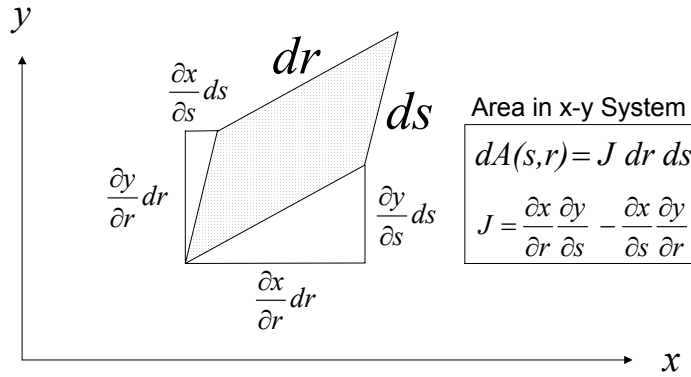


Figure 5.3 True Area in Natural Reference System

5.6 NUMERICAL INTEGRATION IN TWO DIMENSIONS

Numerical integration in two dimensions can be performed using the one-dimensional formulas summarized in Table 5.1. Or:

$$I = \int_{-1}^1 \int_{-1}^1 f(r,s) \mathcal{J}(r,s) \, dr \, ds = \sum_i \sum_j W_i W_j f(r_i, s_j) \mathcal{J}(r_i, s_j) \quad (5.18)$$

Note that the sum of the weighting factors, $W_i W_j$, equals four, the natural area of the element. Most computer programs use 2 by 2 or 3 by 3 numerical

integration formulas. The fundamental problem with this approach is that for certain elements, the 3 by 3 produces elements that are too stiff and the 2 by 2 produces stiffness matrices that are unstable, or, *rank deficient* using matrix analysis terminology. Using a 2 by 2 formula for a nine-node element produces three zero energy displacement modes in addition to the three zero energy rigid body modes. One of these zero energy modes is shown in Figure 5.4.

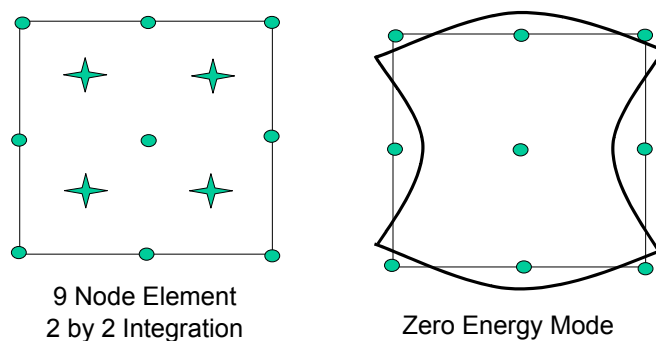


Figure 5.4 A Zero Energy Hourglass Displacement Mode

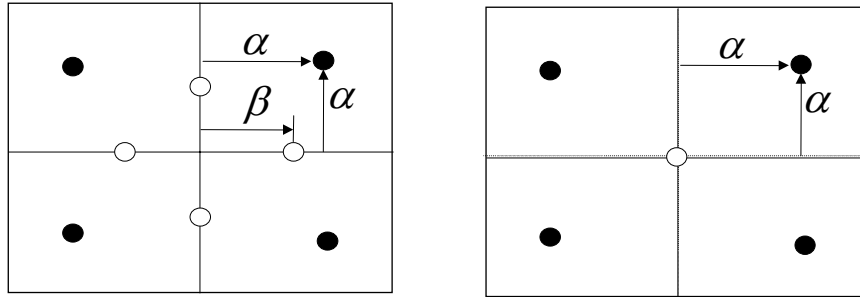
For certain finite element meshes, these zero energy modes may not exist after the element stiffness matrices have been added and boundary conditions applied. In many cases, however, inaccurate results may be produced if reduced integration is used for solid elements. Because of those potential problems, the author recommends the use of true two-dimensional numerical integration methods that are accurate and are always more numerically efficient. Therefore, Equation (5.18) can be written as

$$I = \int_{-1}^1 \int_{-1}^1 f(r,s) \mathcal{J}(r,s) dr ds = \sum_i W_i f(r_i, s_i) \mathcal{J}(r_i, s_i) \quad (5.19)$$

Eight- and five-point formulas exist and are summarized in Figure 5.5.

If $W_\alpha = 9/49$, the eight-point formula gives the same accuracy as the 3 by 3 Gauss product rule, with less numerical effort. On the other hand, if $W_\alpha = 1.0$ the eight-point formula reduces to the 2 by 2 Gauss product rule. If one wants to have the benefits of reduced integration, without the introduction of zero energy

modes, it is possible to let $W_\alpha = 0.99$. Note that the sum of the weight factors equals four.



$$W_\alpha = ?$$

$$W_\beta = 1.0 - W_\alpha$$

$$\alpha = \sqrt{\frac{1.0}{3\sqrt{W_\alpha}}}$$

$$\beta = \sqrt{\frac{2 - 2\sqrt{W_\alpha}}{3W_\beta}}$$

$$W_0 = ?$$

$$W_\alpha = 1.0 - W_0 / 4$$

$$\alpha = \sqrt{\frac{1.0}{3W_\alpha}}$$

Figure 5.5 Eight- and Five-Point Integration Rules

The five point formula is very effective for certain types of elements. It has the advantage that the center point, which in my opinion is the most important location in the element, can be assigned a large weight factor. For example, if W_0 is set to 224/81, the other four integration points are located at $\alpha = \pm\sqrt{0.6}$, with weights of $W_i = 5/9$, which are the same corner points as the 3 by 3 Gauss rule. If W_0 is set to zero, the five-point formula reduces to the 2 by 2 Gauss rule.

5.7 THREE-DIMENSIONAL SHAPE FUNCTIONS

One can easily extend the two-dimensional approach, used to develop the 4- to 9-node element, to three dimensions and create an 8- to 27-node solid element, as shown in Figure 5.6.

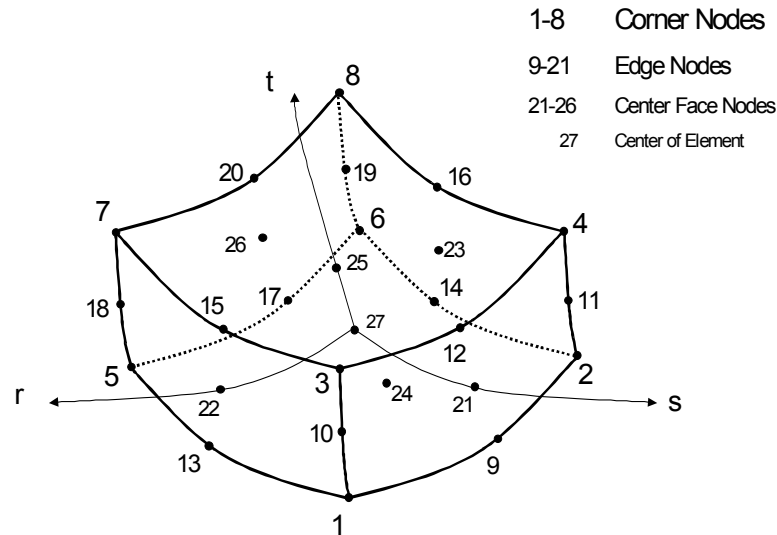


Figure 5.6 Eight- to 27-Node Solid Element

Three-dimensional shape functions are products of the three basic one-dimensional functions and can be written in the following form:

$$G(r_i, s_i, t_i) = g(r, r_i) g(s, s_i) g(t, t_i) \quad (5.20)$$

The terms r_i , s_i and t_i are the natural coordinates of node “ i .” The one-dimensional functions in the r , s and t direction are defined as:

$$\begin{aligned} g_i &= g(r, r_i) = \frac{1}{2}(1 + r_i r) \quad \text{if } r_i = \pm 1 \\ g_i &= g(r, r_i) = (1 + r^2) \quad \text{if } r_i = 0 \\ g_i &= 0 \quad \text{if node } i \text{ does not exist} \end{aligned} \quad (5.21)$$

Using this notation, it is possible to program a shape function subroutine directly without any additional algebraic manipulations. The fundamental requirement of a shape function is that it has a value of 1.0 at the node and is zero at all other nodes. The node shape function is the basic node shape function g_i corrected to be zero at all nodes by a fraction of the basic shape functions at adjacent nodes.

The shape functions N_1 and N_8 for the 8-corner nodes are:

$$N_i = g_i - g_E / 2 - g_F / 4 - g_{27} / 8 \quad (5.22a)$$

The shape functions N_9 and N_{20} for the 12-edge nodes are:

$$N_i = g_i - g_F / 2 - g_{27} / 4 \quad (5.22b)$$

The shape functions N_{21} and N_{26} for the 6 center nodes of each face are:

$$N_i = g_i - g_{27} / 2 \quad (5.22c)$$

The shape function for the node at the center of the element is:

$$N_{27} = g_{27} \quad (5.22d)$$

The term g_E is the sum of the g values at the three adjacent edges. The term g_F is the sum of the g values at the center of the three adjacent faces.

The 27-node solid element is not used extensively in the structural engineering profession. The major reason for its lack of practical value is that almost the same accuracy can be obtained with the 8-node solid element, with the addition of corrected incompatible displacement modes, as presented in the next chapter. The numerical integration can be 3 by 3 by 3 or 2 by 2 by 2 as previously discussed. A nine-point, third-order, numerical integration formula can be used for the eight-node solid element with incompatible modes and, is given by:

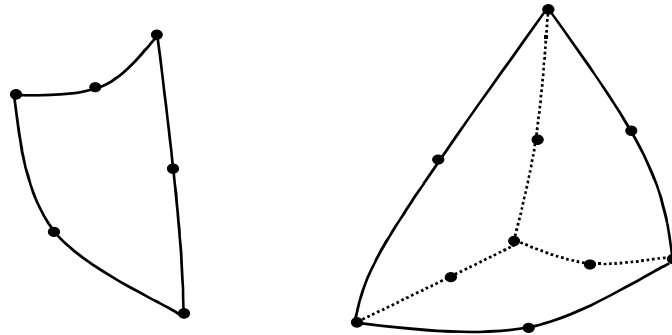
$$W_0 = ?, \quad W_\alpha = 1 - W_0 / 8 \quad \text{and} \quad \alpha = \sqrt{\frac{1}{3W_\alpha}} \quad (5.23)$$

The eight integration points are located at $r = \pm\alpha$, $s = \pm\alpha$ and $t = \pm\alpha$ and the center point is located at the center of the element. If $W_0 = 0$ the formula reduces to the 2 by 2 by 2. If $W_0 = 16/3$ the other eight integration points are located at eight nodes of the element, $\alpha = \pm 1$ and $W_\alpha = 1/3$.

5.8 TRIANGULAR AND TETRAHEDRAL ELEMENTS

The constant strain plane triangular element and the constant strain solid tetrahedral element should never be used to model structures. They are

numerically inefficient, compared to the computational requirements of higher order elements, and do not produce accurate displacements and stresses. However, the six-node plane triangular element and the ten-node solid tetrahedral element, shown in Figure 5.7, are accurate and numerically efficient. The reason for their success is that their shape functions are complete second order polynomials.



A. SIX-NODE TRIANGLE

B. TEN-NODE TETRAHEDRAL

Figure 5.7 Six-Node Plane Triangle and Ten-Node Solid Tetrahedral Elements

They are used extensively for computer programs with special mesh generation or automatic adaptive mesh refinement. They are best formulated in area and volume coordinate systems. For the details and basic formulation of these elements see Cook [5].

5.9 SUMMARY

The use of isoparametric, or natural, reference systems allows the development of curved, higher-order elements. Numerical integration must be used to evaluate element matrices because closed form solutions are not possible for non-rectangular shapes. Elements must have the appropriate number of rigid-body displacement modes. Additional zero energy modes may cause instabilities and oscillations in the displacements and stresses. Constant strain triangular and tetrahedral elements should not be used because of their inability to capture stress gradients. The six-node triangle and ten-node tetrahedral elements produce excellent results.

5.10 REFERENCES

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