18.

FAST NONLINEAR ANALYSIS

The Dynamic Analysis of a Structure with a Small Number of Nonlinear Elements is Almost as Fast as a Linear Analysis

18.1 INTRODUCTION

The response of real structures when subjected to a large dynamic input often involves significant nonlinear behavior. In general, nonlinear behavior includes the effects of large displacements and/or nonlinear material properties.

The use of geometric stiffness and P-Delta analyses, as summarized in Chapter 11, includes the effects of first order large displacements. If the axial forces in the members remain relatively constant during the application of lateral dynamic displacements, many structures can be solved directly without iteration.

The more complicated problem associated with large displacements, which cause large strains in all members of the structure, requires a tremendous amount of computational effort and computer time to obtain a solution. Fortunately, large strains very seldom occur in typical civil engineering structures made from steel and concrete materials. Therefore, the solution methods associated with the large strain problem will not be discussed in detail in this chapter. However, certain types of large strains, such as those in rubber base isolators and gap elements, can be treated as a lumped nonlinear element using the Fast Nonlinear Analysis (FNA) method presented in this chapter.

The more common type of nonlinear behavior is when the material stress-strain, or force-deformation, relationship is nonlinear. This is because of the modern design philosophy that "a well-designed structure should have a limited number of members which require ductility and that the failure mechanism be clearly defined." Such an approach minimizes the cost of repair after a major earthquake.

18.2 STRUCTURES WITH A LIMITED NUMBER OF NONLINEAR ELEMENTS

A large number of very practical structures have a limited number of points or members in which nonlinear behavior takes place when subjected to static or dynamic loading. Local buckling of diagonals, uplifting at the foundation, contact between different parts of the structures and yielding of a few elements are examples of structures with local nonlinear behavior. For dynamic loads, it is becoming common practice to add concentrated damping, base isolation and other energy dissipation elements. Figure 18.1 illustrates typical nonlinear problems. In many cases, those nonlinear elements are easily identified. For other structures, an initial elastic analysis is required to identify the nonlinear areas.

In this chapter the FNA method is applied to both the static and dynamic analysis of linear or nonlinear structural systems. A limited number of predefined nonlinear elements are assumed to exist. Stiffness and mass orthogonal Load Dependent Ritz Vectors of the elastic structural system are used to reduce the size of the nonlinear system to be solved. The forces in the nonlinear elements are calculated by iteration at the end of each time or load step. The uncoupled modal equations are solved exactly for each time increment.

Several examples are presented that illustrate the efficiency and accuracy of the method. The computational speed of the new FNA method is compared with the traditional "brute force" method of nonlinear analysis in which the complete equilibrium equations are formed and solved at each increment of load. For many problems, the new method is several magnitudes faster.

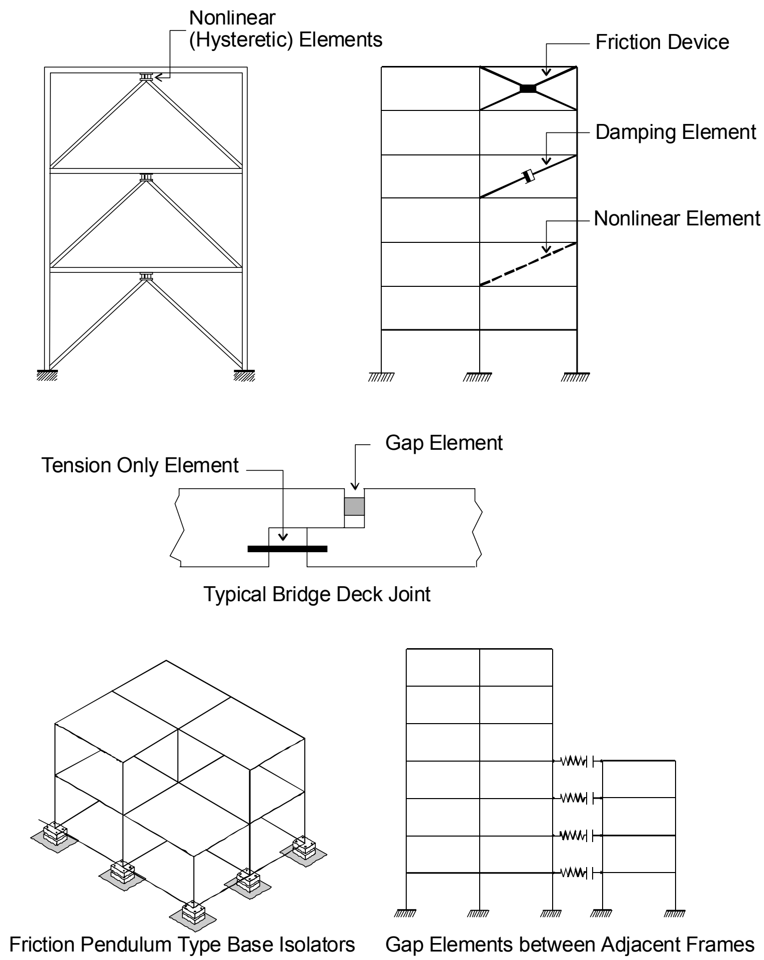

Figure 18.1 Examples of Nonlinear Elements

18.3 FUNDAMENTAL EQUILIBRIUM EQUATIONS

The FNA method is a simple approach in which the fundamental equations of mechanic (equilibrium, force-deformation and compatibility) are satisfied. The ***exact*** force equilibrium of the computer model of a structure at time t is expressed by the following matrix equation:

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)+\mathbf{R}(t)_{NL}=\mathbf{R}(t) \tag{18.1}$$

where $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are the mass, proportional damping and stiffness matrices, respectively. The size of these three square matrices is equal to the total number of unknown node point displacements N_d. The elastic stiffness matrix $\mathbf{K}$ neglects the stiffness of the nonlinear elements. The time-dependent vectors $\ddot{\mathbf{u}}(t), \dot{\mathbf{u}}(t), \mathbf{u}(t)$ and $\mathbf{R}(t)$ are the node point acceleration, velocity, displacement and external applied load, respectively. And $\mathbf{R}(t)_{\mathrm{NL}}$ is the global node force vector from the sum of the forces in the nonlinear elements and is computed by iteration at each point in time.

If the computer model is unstable without the nonlinear elements, one can add "effective elastic elements" (at the location of the nonlinear elements) of arbitrary stiffness. If these effective forces, $\mathbf{K_e u}(t)$, are added to both sides of Equation (1), the exact equilibrium equations can be written as:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + (\mathbf{K} + \mathbf{K_e})\mathbf{u}(t) = \mathbf{R}(t) - \mathbf{R}(t)_{\mathrm{NL}} + \mathbf{K_e}\mathbf{u}(t) \tag{18.2}$$

where $\mathbf{K_e}$ is the effective stiffness of arbitrary value. Therefore, the ***exact*** dynamic equilibrium equations for the nonlinear computer model can be written as:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \overline{\mathbf{K}}\mathbf{u}(t) = \overline{\mathbf{R}}(t) \tag{18.3}$$

The elastic stiffness matrix $\overline{\mathbf{K}}$ is equal to $\mathbf{K} + \mathbf{K_e}$ and is known. The effective external load $\overline{\mathbf{R}}(t)$ is equal to $\mathbf{R}(t) - \mathbf{R}(t)_{\mathrm{NL}} + \mathbf{K_e}\mathbf{u}(t)$, which must be evaluated by iteration. If a good estimate of the effective elastic stiffness can be made, the rate of convergence may be accelerated because the unknown load term $-\mathbf{R}(t)_{\mathrm{NL}} + \mathbf{K_e}\mathbf{u}(t)$ will be small.

18.4 CALCULATION OF NONLINEAR FORCES

At any time the L nonlinear deformations $\mathbf{d}(t)$ within the nonlinear elements are calculated from the following displacement transformation equation:

$$\mathbf{d}(t) = \mathbf{b}\mathbf{u}(t) \tag{18.4}$$

Also, the rate of change with respect to time in the nonlinear deformations, $\dot{\mathbf{d}}(t)$, are given by:

$$\dot{\mathbf{d}}(t) = \mathbf{b}\dot{\mathbf{u}}(t) \tag{18.5}$$

Note that for small displacements, the displacement transformation matrix $\mathbf{b}$ is not a function of time and is ***exact***. The displacement transformation matrix $\mathbf{b}$ for a truss element is given by Equation (2.11).

If the time-history deformations and velocities in all nonlinear elements are known, the nonlinear forces $\mathbf{f}(t)$ in the nonlinear elements can be calculated ***exactly*** at any time from the nonlinear material properties of each nonlinear element. It is apparent that this can only be accomplished by iteration at each point in time.

18.5 TRANSFORMATION TO MODAL COORDINATES

The first step in the solution of Equation (18.3) is to calculate a set of N orthogonal Load Dependent Ritz vectors, Φ, which satisfy the following equations:

$$\Phi^{T}\mathbf{M}\Phi = \mathbf{I} \quad \text{and} \quad \Phi^{T}\overline{\mathbf{K}}\Phi = \Omega^2 \qquad \text{(18.6a) and (18.6b)}$$

where $\mathbf{I}$ is a unit matrix and Ω^2 is a diagonal matrix in which the diagonal terms are defined as ω^2_n.

The response of the system can now be expressed in terms of those vectors by introducing the following matrix transformations:

$$\mathbf{u}(t) = \Phi\mathbf{Y}(t) \qquad \dot{\mathbf{u}}(t) = \Phi\dot{\mathbf{Y}}(t) \qquad \ddot{\mathbf{u}}(t) = \Phi\ddot{\mathbf{Y}}(t) \tag{18.7}$$

The substitution of those equations into Equation (18.1) and the multiplication of both sides of the equation by $\Phi^{\mathbf{T}}$ yield a set of N uncoupled equations expressed by the following matrix equation:

$$\mathbf{I}\ddot{\mathbf{Y}}(t) + \Lambda\dot{\mathbf{Y}}(t) + \Omega^{2}\mathbf{Y}(t) = \mathbf{F}(t) \tag{18.8}$$

in which the linear and nonlinear modal forces are given by:

$$\mathbf{F}(t) = \Phi^{T}\overline{\mathbf{R}}(t) = \Phi^{T}\mathbf{R}(t) - \Phi^{T}\mathbf{R}(t)_{NL} + \Phi^{T}\mathbf{K}_{\mathbf{e}}\mathbf{u}(t) \tag{18.9}$$

The assumption that the damping matrix can be diagonalized is consistent with the classical normal mode superposition method in which damping values are assigned, in terms of percent of critical damping, at the modal level. The diagonal terms of the Λ matrix are $2\xi_n\omega_n$ in which ξ_n is the damping ratio for mode *n*. It should be noted that the forces associated with concentrated dampers at any location in the structure can be included as part of the nonlinear force vector.

Also, if the number of LDR vectors used is equal to the total number of degrees of freedom N_d, Equation 18.8 is exact at time *t*. Therefore, if very small time steps are used and iteration is used within each time step, the method converges to the ***exact*** solution. The use of LDR vectors significantly reduces the number of modes required.

Because $\mathbf{u}(t)=\Phi\mathbf{Y}(t)$, the deformations in the nonlinear elements can be expressed directly in terms of the modal coordinate as:

$$\mathbf{d}(t)=\mathbf{B}\mathbf{Y}(t) \tag{18.10}$$

where the *element deformation* - *modal coordinate* transformation matrix is defined by:

$$\mathbf{B}=\mathbf{b}\Phi \tag{18.11}$$

It is very important to note that the L by N **B** matrix is not a function of time and is relatively small in size; also, it needs to be calculated only once before integration of the modal equations.

At any time, given the deformations and history of behavior in the nonlinear elements, the forces in the nonlinear elements $\mathbf{f}(t)$ can be evaluated from the basic nonlinear properties and deformation history of the element. From the basic principle of virtual work, the nonlinear modal forces are then calculated from:

$$\mathbf{F}(t)_{\mathrm{NL}}=\mathbf{B}^{\mathrm{T}}\mathbf{f}(t) \tag{18.12}$$

The effective elastic forces can also be rewritten as:

$$\mathbf{F}(t)_{\mathrm{e}}=\Phi^{\mathrm{T}}\mathbf{K_e}\mathbf{u}(t)=\Phi^{\mathrm{T}}\mathbf{b}^{\mathrm{T}}\mathbf{k}_{\mathrm{e}}\mathbf{b}\mathbf{u}(t)=\mathbf{B}^{T}\mathbf{k}_{e}\mathbf{d}(t) \tag{18.13}$$

where $\mathbf{k}_e$ is the effective linear stiffness matrix in the local nonlinear element reference system.

18.6 SOLUTION OF NONLINEAR MODAL EQUATIONS

The calculation of the Load Dependent Vectors, without the nonlinear elements, is the first step before solving the modal equations. Also, the **B** deformation-modeshape transformation matrix needs to be calculated only once before start of the step-by-step solution phase. A typical modal equation is of the form:

$$\ddot{y}(t)_n + 2\xi_n \omega_n \dot{y}(t)_n + \omega_n^2 y(t)_n = \overline{f}(t)_n \tag{18.14}$$

where $\bar{f}(t)_n$ is the modal load and for nonlinear elements is a function of all other modal responses at the same point in time. Therefore, the modal equations must be integrated simultaneously and iteration is necessary to obtain the solution of all modal equations at time *t*. The exact solution of the modal equations for a linear or cubic variation of load within a time step is summarized by Equation (13.13) and is in terms of exponential, square root, sine and cosine functions. However, those computational intensive functions, given in Table 13.2, are pre-calculated for all modes and used as constants for the integration within each time step. In addition, the use of the exact piece-wise integration method allows the use of larger time steps.

The complete nonlinear solution algorithm, written in iterative form, is summarized in Table 18.1.

Table 18.1 Summary of Nonlinear Solution Algorithm

I INITIAL CALCULATION - BEFORE STEP-BY-STEP SOLUTION

1. Calculate *N* Load Dependent Ritz vectors Φ for the structure without the nonlinear elements. These vectors have N_d displacement DOF.
2. Calculate the *L* by *N* $\mathbf{B}$ matrix. Where *L* is the total number of DOF within all nonlinear elements.
3. Calculate integration constants $A_1 ---$ for the piece-wise exact integration of the modal equations for each mode.

II NONLINEAR SOLUTION at times $\Delta t, 2\Delta t, 3\Delta t$ --------

1. Use Taylor series to estimate solution at time t .

$$\mathbf{Y}(t) = \mathbf{Y}(t-\Delta t) + \Delta t\dot{\mathbf{Y}}(t-\Delta t) + \frac{\Delta t}{2}\ddot{\mathbf{Y}}(t-\Delta t)$$

$$\dot{\mathbf{Y}}(t) = \dot{\mathbf{Y}}(t-\Delta t) + \Delta t\ddot{\mathbf{Y}}(t-\Delta t)$$

2. For iteration *i,* calculate *L* nonlinear deformations and velocities.

$$\mathbf{d}(t)^i = \mathbf{B}\mathbf{Y}(t)^i \qquad \text{and} \qquad \dot{\mathbf{d}}(t)^i = \mathbf{B}\dot{\mathbf{Y}}(t)^i$$

3. Based on the deformation and velocity histories in nonlinear elements, calculate *L* nonlinear forces $\mathbf{f}(t)^i$.
4. Calculate new modal force vector $\overline{\mathbf{F}}(t)^i = \mathbf{F}(t) - \mathbf{B}^T[\mathbf{f}(t)^i - \mathbf{k}_e\mathbf{d}(t)]$
5. Use piece-wise exact method to solve modal equations for next iteration.

$$\mathbf{Y}(t)^i , \dot{\mathbf{Y}}(t)^i , \ddot{\mathbf{Y}}(t)^i$$

6. Calculate error norm:

$$Err = \frac{\sum_{n=1}^{N} |\overline{f}(t)_n^i| - \sum_{n=1}^{N} |\overline{f}(t)_n^{i-1}|}{\sum_{n=1}^{N} |\overline{f}(t)_n^i|}$$

7. Check Convergence – where the tolerance, Tol , is specified.

If $Err > Tol$ go to step 2 with $i = i + 1$

If $Err < Tol$ go to step 1 with $t = t + \Delta t$

18.7 STATIC NONLINEAR ANALYSIS OF FRAME STRUCTURE

The structure shown in Figure 18.2 is used to illustrate the use of the FNA algorithm for the solution of a structure subjected to both static and dynamic loads. It is assumed that the external columns of the seven-story frame structure cannot take axial tension or moment at the foundation level and the column can uplift. The axial foundation stiffness is 1,000 kips per inch at the external columns and 2,000 kips per inch at the center column. The dead load is 80 kips per story and is applied as concentrated vertical loads of 20 kips at the external columns and 40 kips at the center column. The static lateral load is specified as 50 percent of the dead load.

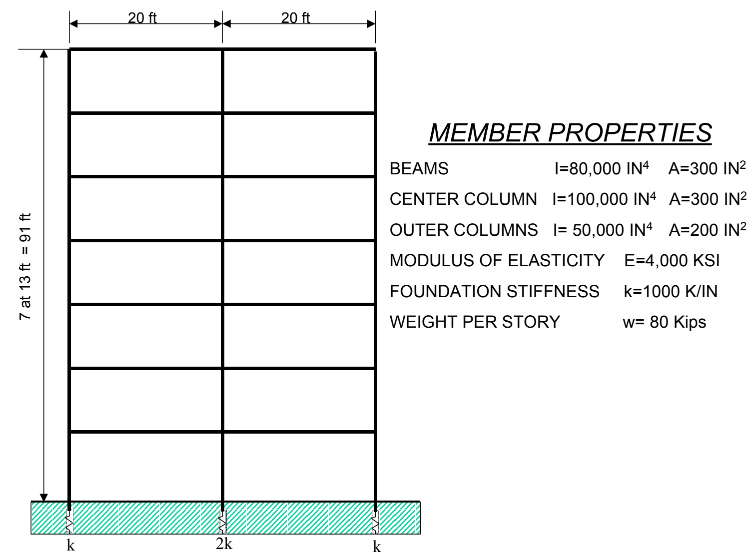

Figure 18.2 Properties of Frame Structure

For the purpose of calculating the dynamic response, the mass of the structure is calculated directly from the dead load. The fundamental period of the structure with the external columns not allowed to uplift is 0.708 seconds. The fundamental period of the structure allowing uplift is 1.691 seconds.

The static load patterns used to generate the series of LDR vectors are shown in Figure 18.3. The first load pattern represents the mass-proportional lateral earthquake load. The second pattern represents the vertical dead load. The last two load patterns represent the possible contact forces that exist at the foundation of the external columns. It is very important that equal and opposite load patterns be applied at each point where a nonlinear element exists. These vectors allow for the accurate evaluation of member forces at the contact points. For this example, the vectors will not be activated in the solution when there is uplift at the base of the columns because the axial force must be zero. Also, the total number of Ritz vectors used should be a multiple of the number of static load patterns so that the solution is complete for all possible loadings. In addition, care should be taken to make sure that all vectors are linearly independent.

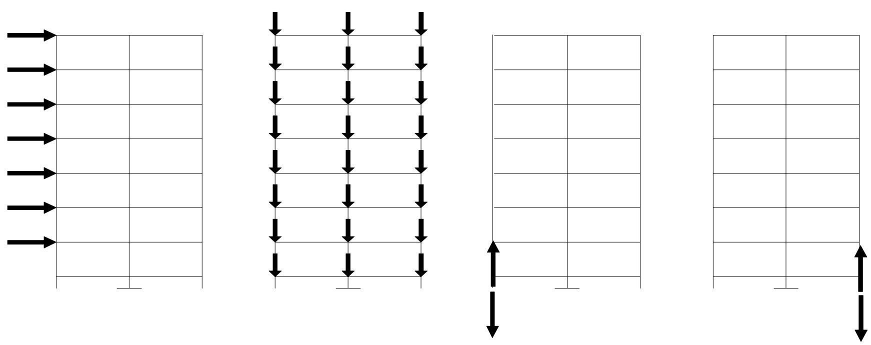

Figure 18.3 Four Static Load Vectors Used in Analysis

For this example, the dead load is applied at time zero and reaches its maximum value at one second, as shown in Figure 18.4. The time increment used is 0.10 second. The modal damping ratios are set to 0.999 for all modes; therefore, the dynamic solution converges to the static solution in less than one second. The lateral load is applied at two seconds and reaches a maximum value at three seconds. At four seconds after 40 load increments, a static equilibrium position is obtained.

It should be noted that the converged solution is the exact static solution for this problem because all possible combinations of the static vectors have been included in the analysis. The magnitude of the mass, damping and the size of the time step used will not affect the value of the converged static solution.

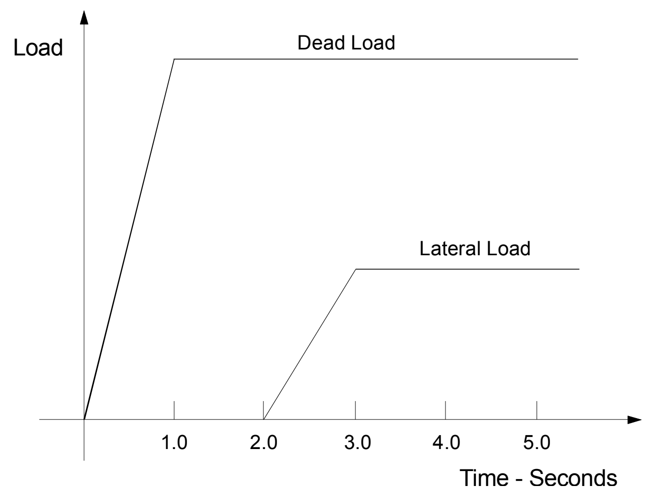

Figure 18.4 Application of Static Loads vs. Time

It is of interest to note that it is impossible for a real structure to fail under static loads only, because at the point of collapse, inertia forces must be present. Therefore, the application of static load increments with respect to time is a physically realistic approach. The approximate static load response of the frame is shown in Figure 18.5.

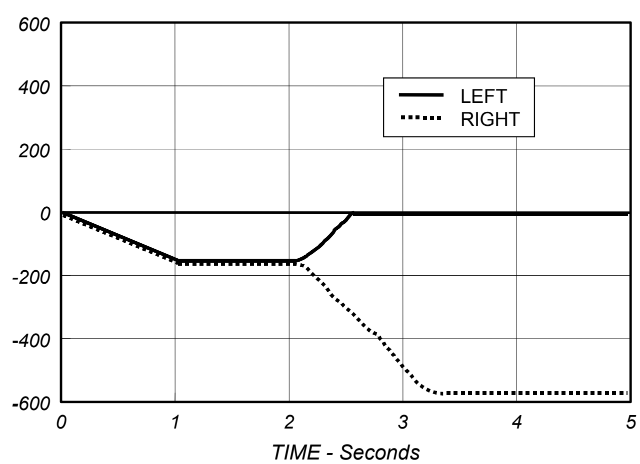

Figure 18.5 Column Axial Forces from "Static" Loads

18.8 DYNAMIC NONLINEAR ANALYSIS OF FRAME STRUCTURE

The same frame structure that is defined in Figure 18.2 is subjected to Loma Prieta Earthquake ground motions recorded on the east side of the San Francisco Bay at a maximum acceleration of 20.1 percent of gravity and a maximum ground displacement of 5.81 inches. The acceleration record used was corrected to zero acceleration, velocity and displacement at the end of the record and is shown in Figure 18.6.

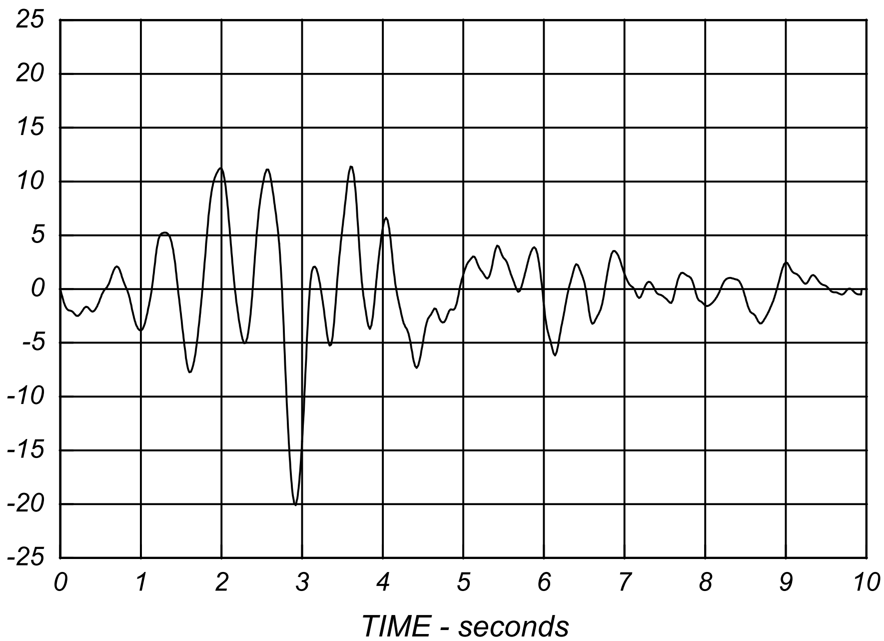

Figure 18.6 Segment of Loma Prieta Earthquake - Percent of Gravity

The dead load was applied as a ramp function in the time interval 0 to 1 second. The lateral earthquake load is applied starting at 2 seconds. Sixteen Ritz vectors and a modal damping value of 5 percent were used in the analysis. The column axial forces as a function of time are shown in Figure 18.7.

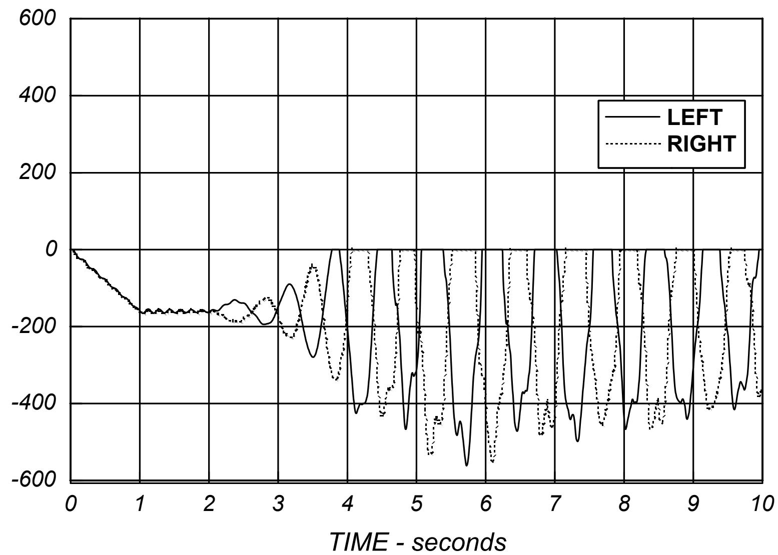

Figure 18.7 Column Axial Forces from Earthquake Loading

It is of considerable interest to compare the behavior of the building that is not allowed to uplift with the behavior of the same building that is allowed to uplift. These results are summarized in Table 18.2.

Table 18.2. Summary of Results for Building Uplifting Problem from the Loma Prieta Earthquake $\xi = 0.05$

Uplift	Max. Displace-ment (inches)	Max. Axial Force (kips)	Max. Base Shear (kips)	Max. Base Moment (k-in)	Max. Strain Energy (k-in)	Compu-tational Time (seconds)
Without	3.88	542	247	212,000	447	14.6
With	3.90	505	199	153,000	428	15
Percent Difference	+0.5 %	-6.8%	-19.4%	-27.8%	-4.2%	+3%

The lateral displacement at the top of the structure has not changed significantly by allowing the external columns to uplift. However, allowing column uplifting reduces significantly the base shear, overturning moment and strain energy stored in the structure. It is apparent for this structure, that uplifting is a “natural” base isolation system. This reduction of forces in a structure from uplifting has also been observed in shaking table tests. However, it has not been used extensively for real structures because of the lack of precedent and the inability of the design engineer to easily compute the dynamic behavior of an uplifting structure.

For this small nonlinear example, there is a very small increase in computational time compared to a linear dynamic analysis. However, for a structural system with a large number of nonlinear elements, a large number of Ritz vectors may be required and the additional time to integrate the nonlinear modal equation can be significant.

Table 18.3 presents a summary of the results if the same structure is subjected to twice the ground accelerations of the Loma Prieta earthquake. One notes that all significant response parameters are reduced significantly.

Table 18.3 Summary of Results for Building Uplifting Problem from Two Times the Loma Prieta Earthquake - $\xi = 0.05$

Uplift	Max. Displace-ment (inches)	Max. Column Force (kips)	Max. Base Shear (kips)	Max. Base Moment (k-in)	Max. Strain Energy (k-in)	Max. Uplift (inches)
Without	7.76	924	494	424,000	1,547	
With	5.88	620	255	197,000	489	1.16
Percent Difference	-24%	-33%	-40%	-53%	-68%	

The maximum uplift at the base of the external columns is more than one inch; therefore, these may be ideal locations for the placement of additional energy dissipation devices such as viscous dampers.

18.9 SEISMIC ANALYSIS OF ELEVATED WATER TANK

A nonlinear earthquake response analysis of an elevated water tank was conducted using a well-known commercial computer program in which the stiffness matrix for the complete structure was recalculated for each time step and equilibrium was obtained using iteration. The structural system and analysis had the following properties:

92 nodes with 236 unknown displacements
103 elastic frame elements
56 nonlinear diagonal brace elements - tension only
600 time steps at 0.02 seconds

The solution times on two different computers are listed below:

Intel 486	3 days	4,320	minutes
Cray XMP-1	3 hours	180	minutes

The same structure was solved using the FNA method presented in this chapter on an Intel 486 in less than 3 minutes. Thus, a structural engineer has the ability to investigate a large number of retrofit strategies within a few hours.

18.10 SUMMARY

It is common practice in engineering design to restrict the nonlinear behavior to a small number of predefined locations within a structure. In this chapter an efficient computational method has been presented to perform the static and dynamic analysis of these types of structural systems. The FNA method, using LDR vectors, is a completely different approach to structural dynamics. The nonlinear forces are treated as external loads and a set of LDR vectors is generated to accurately capture the effects of those forces. By iteration within each time step, *equilibrium*, *compatibility* and all element *force-deformation* equations within each nonlinear element are identically satisfied. The reduced set of modal equations is solved exactly for a linear variation of forces during a small time step. Numerical damping and integration errors from the use of large time steps are not introduced.

The computer model must be structurally stable without the nonlinear elements. All structures can be made stable if an element with an effective stiffness is placed parallel with the nonlinear element and its stiffness added to the basic computer model. The forces in this effective stiffness element are moved to the right side of the equilibrium equations and removed during the nonlinear iterative solution phase. These dummy or effective stiffness elements will eliminate the introduction of long periods into the basic model and improve accuracy and rate of convergence for many nonlinear structures.

It has been demonstrated that structures subjected to static loads can also be solved using the FNA method. It is only necessary to apply the loads slowly to a constant value and add large modal damping values. Therefore, the final converged solution will be in static equilibrium and will not contain inertia forces. It should be noted that it is necessary to use Load Dependent Vectors associated with the nonlinear degrees of freedom, and not the exact eigenvectors, if static problems are to be solved using this approach.

The FNA method has been added to the commercial program ETABS for the analysis of building systems and the general purpose structural analysis program SAP2000. The ETABS program has special base isolation elements that are commonly used by the structural engineering profession. Those computer programs calculate and plot the total input energy, strain energy, kinetic energy and the dissipation of energy by modal damping and nonlinear elements as a function of time. In addition, an energy error is calculated that allows the user to evaluate the appropriate time step size. Therefore, the energy calculation option allows different structural designs to be compared. In many cases a good design for a specified dynamic loading is one that has a minimum amount of strain energy absorbed within the structural system.

As in the case of normal linear mode superposition analysis, it is the responsibility of the user to check, using multiple analyses, that a sufficiently small time step and the appropriate number of modes have been used. This approach will ensure that the method will converge to the exact solution.

Using the numerical methods presented in this chapter, the computational time required for a nonlinear dynamic analysis of a large structure, with a small number of nonlinear elements, can be only a small percentage more than the computational time required for a linear dynamic analysis of the same structure. This allows large nonlinear problems to be solved relatively quickly.