

3.

ENERGY AND WORK

All External Work Supplied to a Real Structural System is Stored or Dissipated as Energy

3.1 INTRODUCTION

A large number of energy methods have been presented during the last 150 years for the analysis of both determinate and statically indeterminate structures. However, if all methods are formulated in matrix notation, it can be shown that only two fundamental methods exist. They are generally defined as the force and displacement methods. One can use minimum energy principles or methods of virtual-work to derive the general equations for linear structural analysis. **Energy** is defined as the ability to do **work**. Both have the units of force-distance.

For many types of structural elements, however, there can be many advantages in using both force and displacement methods in approximating the stiffness properties of the element. For example, the classical non-prismatic beam element uses a force approach to define the forces at a typical cross-section within the beam; however, a displacement approximation, such as plane sections remain plane, is used to define the strain distribution over the cross-section.

In recent years, assumed-stress hybrid formulations have been used to produce element stiffness properties. In addition, assumed-stress distributions, virtual work methods and the least-square error approach have been used to calculate accurate stresses in displacement-based finite elements. Therefore, no one method can be used to solve all problems in structural analysis. The only

restriction on the computational techniques used is that the results must converge to the exact values as the elements become smaller.

3.2 VIRTUAL AND REAL WORK

The principles of virtual work are very simple and are clear statements of conservation of energy. The principles apply to structures that are in equilibrium in a real displaced position \mathbf{u} when subjected to loading \mathbf{R} . The corresponding real internal deformations and internal forces are \mathbf{d} and \mathbf{f} respectively. All terms are illustrated in Figures 3.1 and 3.2.

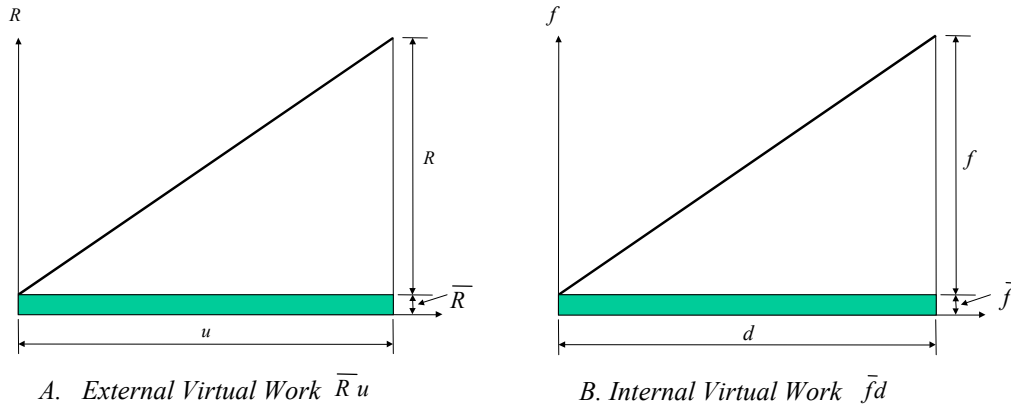


Figure 3.1 Method of Virtual Forces

The principle of virtual forces states (in my words) *if a set of infinitesimal external forces, $\bar{\mathbf{R}}$, in equilibrium with a set of infinitesimal internal forces $\bar{\mathbf{f}}$ that exist before the application of the real loads and displacements, the external virtual work is equal to the internal virtual work.* Or, in terms of the notation defined previously:

$$\bar{\mathbf{R}}^T \mathbf{u} = \bar{\mathbf{f}}^T \mathbf{d} \quad (3.1)$$

If only one joint displacement u_i is to be calculated, only one external virtual load exists, $\bar{R}_i = 1$. For this case, the equation is the same as the unit load method. It is apparent for nonlinear analysis that the principle of virtual forces

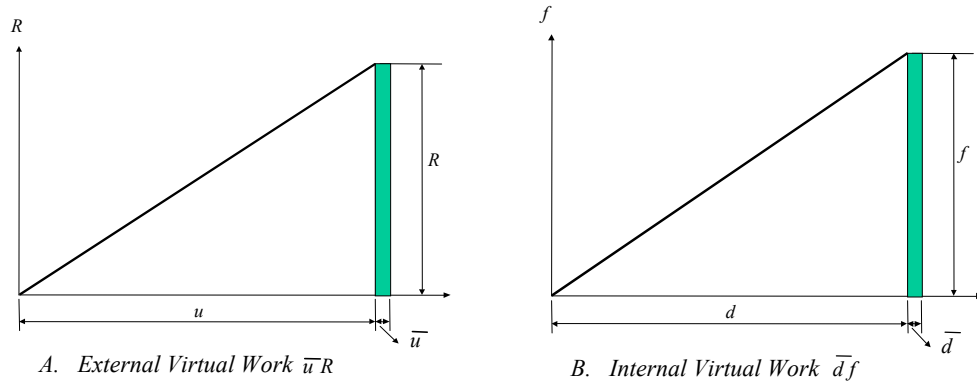


Figure 3.2 Method of Virtual Displacements

cannot be used, because the linear relationship between $\bar{\mathbf{R}}$ and $\bar{\mathbf{f}}$ may not hold after the application of the real loads and displacements.

The principle of virtual displacements states (in my words) *if a set of infinitesimal external displacements, $\bar{\mathbf{u}}$, consistent with a set of internal virtual displacements, $\bar{\mathbf{d}}$, and boundary conditions are applied after the application of the real loads and displacements, the external virtual work is equal to the internal virtual work.* Or, in terms of matrix notation:

$$\bar{\mathbf{u}}^T \mathbf{R} = \bar{\mathbf{d}}^T \mathbf{f} \quad (3.2)$$

It is important to note that the principle of virtual displacements does apply to the solution of nonlinear systems because the virtual displacements are applied to real forces in the deformed structure.

In the case of finite element analysis of continuous solids, the virtual work principles are applied at the level of stresses and strains; therefore, integration over the volume of the element is required to calculate the virtual work terms.

For linear analysis, it is apparent that the real external work, or energy, is given by:

$$W_E = \frac{1}{2} \mathbf{u}^T \mathbf{R} = \frac{1}{2} \mathbf{R}^T \mathbf{u} \quad (3.3)$$

The real internal work, or strain energy, is given by:

$$W_I = \frac{1}{2} \mathbf{d}^T \mathbf{f} = \frac{1}{2} \mathbf{f}^T \mathbf{d} \quad (3.4)$$

3.3 POTENTIAL ENERGY AND KINETIC ENERGY

One of the most fundamental forms of energy is the position of a mass within a gravitational field near the earth's surface. The gravitational potential energy V_g is defined as the constant weight w moved against a constant gravitational field of distance h . Or:

$$V_g = mgh \quad \text{or} \quad V_g = Wh \quad (3.5)$$

A mass that is moving with velocity v has kinetic energy given by the following equation:

$$V_k = \frac{1}{2} mv^2 \quad (3.6)$$

One of the most common examples that illustrates the physical significance of both the potential and kinetic energy is the behavior of a pendulum shown in Figure 3.3.

If the mass of the pendulum has an initial position of h_{\max} , the kinetic energy is zero and the potential energy is $h_{\max}W$. When h equals zero, the potential energy is zero; therefore, from conservation of energy, the kinetic energy is:

$$V_k = h_{\max} W = \frac{Wv^2}{2g} \quad (3.7)$$

Hence, the maximum horizontal velocity is:

$$v_{\max} = \sqrt{2g h_{\max}} \quad (3.8)$$

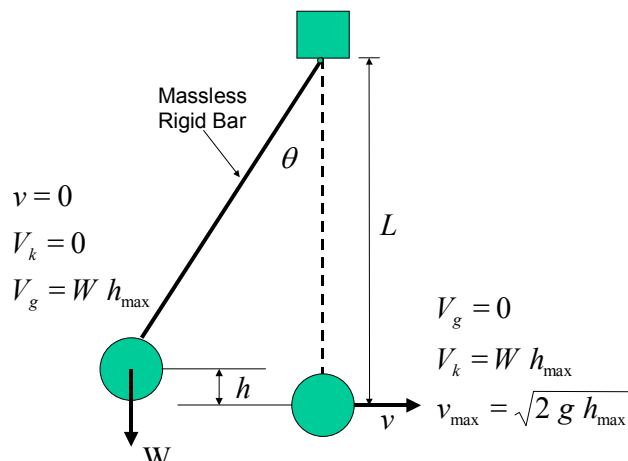


Figure 3.3 Oscillation of Pendulum

It is important to note that the total energy in the oscillating system is always constant; therefore, the following energy equation, at any time t , must be satisfied:

$$V_g(t) + V_k(t) = W h_{\max} = \text{constant} \quad (3.9)$$

The physical behavior of the oscillating pendulum can be considered to be an **energy pump**, where there is an interchange between potential and kinetic energy.

The tangential force accelerating the mass is $W \sin \theta$. From Newton's Second Law, the following nonlinear, differential equation of equilibrium can be written:

$$mL\ddot{\theta} + W \sin \theta = 0 \quad \text{or,} \quad \ddot{\theta} + \frac{g}{L} \sin \theta = 0 \quad (3.10)$$

For very small angles, $\sin \theta \approx \theta$, the approximate linear differential equation is:

$$\ddot{\theta} + \frac{g}{L} \theta = 0 \quad (3.11)$$

Hence, the *small displacement* period of oscillation of a pendulum is:

$$T = 2\pi \sqrt{\frac{L}{g}} \quad (3.12)$$

3.4 STRAIN ENERGY

The strain energy stored in an element "i" within a general structural system is the area under the stress-strain diagram integrated over the volume of the element. For linear systems, the stress-strain matrix $\mathbf{E}^{(i)}$, including initial thermal stresses $\mathbf{f}_t^{(i)}$, can be written in matrix form as:

$$\mathbf{f}^{(i)} = \mathbf{E}^{(i)} \mathbf{d}^{(i)} + \mathbf{f}_t^{(i)} \quad (3.13)$$

The column matrices $\mathbf{f}^{(i)}$ and $\mathbf{d}^{(i)}$ are the stresses and strain respectively. Therefore, the strain energy within one element is given by:

$$W_I^{(i)} = \frac{1}{2} \int \mathbf{d}^{(i)T} \mathbf{E}^{(i)} \mathbf{d}^{(i)} dV + \int \mathbf{d}^{(i)T} \mathbf{f}_t^{(i)} dV \quad (3.14)$$

Within each element, an approximation can be made on the displacements. Or:

$$u_x^{(i)} = \mathbf{N}^{(i)} \mathbf{u}_x, \quad u_y^{(i)} = \mathbf{N}^{(i)} \mathbf{u}_y \quad \text{and} \quad u_z^{(i)} = \mathbf{N}^{(i)} \mathbf{u}_z \quad (3.15)$$

Hence, after the application of the strain-displacement equations, the element strains can be expressed in terms of nodal displacements. Or:

$$\mathbf{d}^{(i)} = \mathbf{B}^{(i)} \mathbf{u} \quad \text{or} \quad \mathbf{d}^{(i)T} = \mathbf{u}^T \mathbf{B}^{(i)T} \quad (3.16)$$

The column matrix \mathbf{u} contains all of the node, or joint, displacements of the complete structural system. In addition, it may contain displacement patterns within the element. When equation (3.16) is written in this form, it is apparent that the $\mathbf{B}^{(i)}$ matrix can be very large; however, it only has non-zero terms associated with the displacements at the nodes connected to nodes adjacent to the element. Therefore, the $\mathbf{B}^{(i)}$ matrix is always formed and used in compacted form within a computer program, and an *integer location array*, $\mathbf{L}_a^{(i)}$, is formed for

each element that is used to relate the local node displacements $\mathbf{u}^{(i)}$ to the *global* node displacements \mathbf{u} .

After integration over the volume of the element, the strain energy, in terms of the global node displacements, can be written as:

$$W_I^{(i)} = \frac{1}{2} \mathbf{u}^T \mathbf{k}^{(i)} \mathbf{u} + \mathbf{u}^T \mathbf{F}_t^{(i)} \quad (3.17)$$

Therefore, the element stiffness matrix is by definition:

$$\mathbf{k}^{(i)} = \int \mathbf{B}^{(i)T} \mathbf{E}^{(i)} \mathbf{B}^{(i)} dV \quad (3.18)$$

And the element thermal force matrix is:

$$\mathbf{F}^{(i)} = \int \mathbf{B}^{(i)T} \mathbf{f}_t^{(i)} dV \quad (3.19)$$

The total internal strain energy is the sum of the element strain energies. Or:

$$W_I = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} + \mathbf{u}^T \mathbf{F}_t \quad (3.20)$$

The global stiffness matrix \mathbf{K} is the sum of the element stiffness matrices $\mathbf{k}^{(i)}$. Or:

$$\mathbf{K} = \sum \mathbf{k}^{(i)} \quad (3.21)$$

The summation of element stiffness matrices to form the global stiffness matrix is termed the *direct stiffness method*. The global thermal load vector \mathbf{F}_t is the sum of the element thermal load matrices:

$$\mathbf{F}_t = \sum \mathbf{F}_t^{(i)} \quad (3.22)$$

3.5 EXTERNAL WORK

The external work W_c performed by a system of concentrated node, or joint, loads \mathbf{F}_c is:

$$W_c = \frac{1}{2} \mathbf{u}^T \mathbf{F}_c \quad (3.23)$$

Within each element "i", the external work $W_g^{(i)}$ performed by the body forces because of gravitational loads is:

$$W_g^{(i)} = \frac{1}{2} \int (m_x^{(i)} g_x u_x^{(i)} + \rho_y g_y u_y + \rho_z g_z u_z) dV \quad (3.24)$$

Application of the displacement assumptions given by Equation (3.15), integration over the volume of the element, and summation over all elements produces the following equation for the energy because of body forces:

$$W_g = \frac{1}{2} \mathbf{u}^T \mathbf{F}_g \quad (3.25)$$

The external work W_s^j performed because of element surface stresses (tractions) $\mathbf{t}_s^{(j)}$, for a typical surface "j" is of the form:

$$W_s^{(j)} = \frac{1}{2} \mathbf{u}^T \int \mathbf{B}_s^{(j)T} \mathbf{t}_s^{(j)} dS \quad (3.26)$$

Application of the displacement assumptions given by Equation (3.15), integration over the surface of the element, and summation over all surface elements produces the following equation for the energy because of surface tractions:

$$W_s = \frac{1}{2} \mathbf{u}^T \mathbf{F}_s \quad (3.27)$$

Therefore, the total external work performed on any system of structural elements is:

$$W_E = \frac{1}{2} \mathbf{u}^T [\mathbf{F}_c + \mathbf{F}_g + \mathbf{F}_s] \quad (3.28)$$

3.6 STATIONARY ENERGY PRINCIPLE

It is a fact for linear systems that the internal strain energy must equal the external work performed on the structure. For a single degree-of-freedom system, we can use this principle to solve for the displacement. However, for a multi degree-of-freedom system, a different approach is required. The energy plots, shown in Figure 3.4, illustrate that a new energy function Ω can be defined.

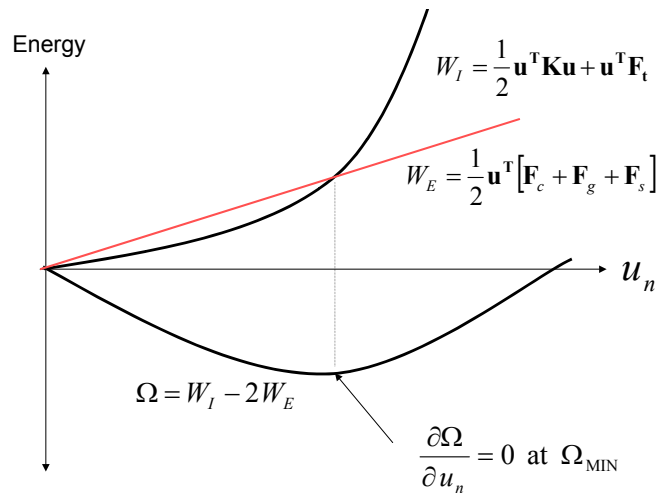


Figure 3.4 Energy as a Function of a Typical Displacement

It is apparent that the solution at the point of minimum potential energy is where the internal energy equals the external energy. Therefore, the major advantage of the use of the potential energy function is that the solution must satisfy the following equation for all displacement degrees-of-freedom u_n :

$$\frac{\partial \Omega}{\partial u_n} = 0 \quad (3.29)$$

The energy function written in matrix form is:

$$\Omega = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{R} \quad (3.30)$$

The resultant load vector \mathbf{R} associated with the four types of loading is:

$$\mathbf{R} = \mathbf{F}_c + \mathbf{F}_g + \mathbf{F}_s - \mathbf{F}_t \quad (3.31)$$

Application of Equation (3.29) to all displacements yields:

$$\begin{bmatrix} \frac{\partial \Omega}{\partial \mathbf{u}_1} \\ - \\ \frac{\partial \Omega}{\partial \mathbf{u}_2} \\ - \\ \frac{\partial \Omega}{\partial \mathbf{u}_n} \\ - \\ \frac{\partial \Omega}{\partial \mathbf{u}_N} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & - & \mathbf{0} & - & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & - & \mathbf{0} & - & \mathbf{0} \\ - & - & - & - & - & - \\ \mathbf{0} & \mathbf{0} & - & \mathbf{1} & - & \mathbf{0} \\ - & - & - & - & - & - \\ \mathbf{0} & \mathbf{0} & - & \mathbf{0} & - & \mathbf{1} \end{bmatrix} [\mathbf{K}\mathbf{u} - \mathbf{R}] = [\mathbf{0}] \quad (3.32)$$

Therefore, the node equilibrium equation for all types of structural systems can be written as the following matrix equation:

$$\mathbf{K}\mathbf{u} = \mathbf{R} \quad (3.33)$$

The only approximation involved in the development of this equation is the assumption of the displacement patterns within each element. If the same displacement approximation is used to calculate the kinetic energy, the resulting mass matrix is termed a *consistent mass matrix*.

Another important fact concerning compatible displacement-based finite elements is that they converge from below, to the exact solution, as the mesh is refined. Therefore, the displacements and stresses tend to be lower than the exact values. From a practical structural engineering viewpoint, this can produce very dangerous results. To minimize this problem, the structural engineer must check statics and conduct parameter studies using different meshes.

3.7 THE FORCE METHOD

The traditional method of *cutting* a statically indeterminate structure, applying redundant forces, and solving for the redundant forces by setting the relative

displacements at the cuts to zero has been the most popular method of structural analysis, if hand calculations are used. The author has developed structural analysis programs based on both the force and displacement methods of analysis. At this point in time, there appears to be no compelling reason for using the force method within a computer program for solving large structural systems. In fact, programs based on the displacement approach are simple to program and, in general, require less computer time to execute. Another significant advantage of a displacement approach is that the method is easily extended to the dynamic response of structures.

To develop the stiffness of one-dimensional elements, however, the force method should be used because the internal forces can be expressed exactly in terms of the forces at the two ends of the element. Therefore, the force method will be presented here for a single-element system.

Neglecting thermal strains, the energy function can be written as:

$$\Omega = \frac{1}{2} \int \mathbf{f}^T \mathbf{d} \, dV - \mathbf{R}^T \mathbf{u} \quad (3.34)$$

The internal forces can be expressed in terms of the node forces using the following equation:

$$\mathbf{f} = \mathbf{P} \mathbf{R} \quad (3.35)$$

For linear material $\mathbf{d} = \mathbf{C} \mathbf{f}$ and the energy function can be written as:

$$\Omega = \frac{1}{2} \mathbf{R}^T \mathbf{F} \mathbf{R} - \mathbf{R}^T \mathbf{u} \quad (3.36)$$

Where the element flexibility matrix is:

$$\mathbf{F} = \int \mathbf{P}^T \mathbf{C} \mathbf{P} \, dV \quad (3.37)$$

We can now minimize the *complementary energy function* by requiring that:

$$\frac{\partial \Omega}{\partial R_n} = 0 \quad (3.38)$$

The node displacements can now be expressed in terms of node forces by:

$$\mathbf{u} = \mathbf{FR} \quad (3.39)$$

The element stiffness can now be numerically evaluated from:

$$\mathbf{K} = \mathbf{F}^{-1} \quad (3.40)$$

The element stiffness can be used in the direct stiffness approach where the basic unknowns are the node displacements. One can also derive the element flexibility by applying the virtual force method.

3.8 LAGRANGE'S EQUATION OF MOTION

In the case of dynamic analysis of structures, the direct application of the well-known Lagrange's equation of motion can be used to develop the dynamic equilibrium of a complex structural system[1]. Lagrange's minimization equation, written in terms of the previously defined notation, is given by:

$$\frac{\partial}{\partial t} \left(\frac{\partial V_k}{\partial \dot{u}_n} \right) - \frac{\partial V_k}{\partial u_n} + \frac{\partial \Omega}{\partial u_n} = 0 \quad (3.41)$$

The node point velocity is defined as \dot{u}_n . The most general form for the kinetic energy $V_k^{(i)}$ stored within a three-dimensional element i of mass density ρ is:

$$V_k^{(i)} = \int \frac{1}{2} \begin{bmatrix} \dot{u}_x & \dot{u}_y & \dot{u}_z \end{bmatrix} \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \dot{u}_x \\ \dot{u}_y \\ \dot{u}_z \end{bmatrix} dV \quad (3.42)$$

The same shape functions used to calculate the strain energy within the element allow the velocities within the element to be expressed in terms of the node point velocities. Or:

$$\dot{u}_x^{(i)} = \mathbf{N}^{(i)} \dot{\mathbf{u}}_x, \quad \dot{u}_y^{(i)} = \mathbf{N}^{(i)} \dot{\mathbf{u}}_y \quad \text{and} \quad \dot{u}_z^{(i)} = \mathbf{N}^{(i)} \dot{\mathbf{u}}_z \quad (3.43)$$

Therefore, the velocity transformation equations can be written in the following form:

$$\begin{bmatrix} \dot{u}_x^{(i)} \\ \dot{u}_y^{(i)} \\ \dot{u}_z^{(i)} \end{bmatrix} = \bar{\mathbf{N}}^{(i)} \dot{\mathbf{u}} \quad (3.44)$$

Using exact or numerical integration, it is now possible to write the total kinetic energy within a structure as:

$$V_k = \sum_i V_k^{(i)} = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} \quad (3.45)$$

The total mass matrix \mathbf{M} is the sum of the element mass matrices $\mathbf{M}^{(i)}$. The element consistent mass matrices are calculated from:

$$\mathbf{M}^{(i)} = \int \bar{\mathbf{N}}^{(i)T} \mathbf{m} \bar{\mathbf{N}}^{(i)} dV \quad (3.46)$$

where \mathbf{m} is the 3 by 3 diagonal mass density matrix shown in Equation (3.42). Equation (3.46) is very general and can be used to develop the *consistent mass matrix* for any displacement-based finite element. The term “consistent” is used because the same shape functions are used to develop both the stiffness and mass matrices.

Direct application of Equation (3.41) will yield the dynamic equilibrium equations:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{R} \quad (3.47)$$

Later in the book the more general dynamic equilibrium equations with damping will be developed using a physical equilibrium approach.

3.9 CONSERVATION OF MOMENTUM

The conservation of momentum is often presented as a fundamental principle of physics. However, it can be easily derived from the basic equilibrium equations. Consider the two rigid bodies shown in Figure 3.5.

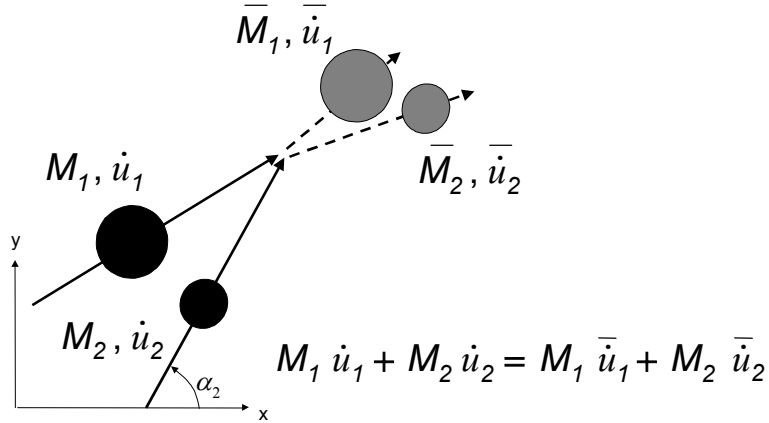


Figure 3.5 Conservation of Linear Momentum

From Newton's Second Law, the equal and opposite forces acting on the rigid bodies during impact will be:

$$F = M\ddot{u} \approx M \frac{\dot{u} - \bar{u}}{\delta t} \quad (3.48)$$

If the duration of contact between the two bodies is δt , the contact force can be approximated by a change in the velocity before and after impact. During contact, equilibrium must be satisfied in both the x and y directions. Therefore:

$$\begin{aligned} F_x \delta t &= M_1(\dot{u}_{1x} - \bar{u}_{1x}) + M_2(\dot{u}_{2x} - \bar{u}_{2x}) = 0 \\ F_y \delta t &= M_1(\dot{u}_{1y} - \bar{u}_{1y}) + M_2(\dot{u}_{2y} - \bar{u}_{2y}) = 0 \end{aligned} \quad (3.49)$$

Momentum is defined as mass times the velocity of the mass and has the properties of a vector. From Equation (3.49), momentum has the direction of the velocity and its components can be plus or minus in reference to the x-y system. Or:

$$\begin{aligned} M_1 \dot{u}_{1x} + M_2 \dot{u}_{2x} &= M_1 \bar{u}_{1x} + M_2 \bar{u}_{2x} \\ M_1 \dot{u}_{1y} + M_2 \dot{u}_{2y} &= M_1 \bar{u}_{1y} + M_2 \bar{u}_{2y} \end{aligned} \quad (3.50)$$

In addition, the resultant momentum vector must be the same before and after impact. Or:

$$M_1 \dot{u}_1 + M_2 \dot{u}_2 = M_1 \bar{u}_1 + M_2 \bar{u}_2 \quad (3.51)$$

It is apparent that three equations, given by Equations (3.50) and (3.51), do not have a unique solution because there are four unknowns. The following principle of conservation of kinetic energy must be enforced as an additional condition:

$$M_1 \dot{u}_1^2 + M_2 \dot{u}_2^2 = M_1 \bar{u}_1^2 + M_2 \bar{u}_2^2 \quad (3.52)$$

Consider a direct collision, with no energy dissipation, of a mass M_1 at a known velocity \dot{u}_1 with a mass of M_2 that is at rest. Conservation of momentum (equilibrium) and conservation of kinetic energy requires that:

$$\begin{aligned} M_1 \dot{u}_1 &= M_1 \bar{u}_1 + M_2 \bar{u}_2 \\ M_1 \dot{u}_1^2 &= M_1 \bar{u}_1^2 + M_2 \bar{u}_2^2 \end{aligned} \quad (3.53)$$

After impact, the new velocities are:

$$\bar{u}_1 = \frac{M_1 - M_2}{M_1 + M_2} \dot{u}_1 \quad \text{and} \quad \bar{u}_2 = \frac{2M_1}{M_1 + M_2} \dot{u}_1 \quad (3.54)$$

If the two masses are equal, the velocity of the first is reduced to zero. If the first mass is less than the second mass, the first will bounce back and the large mass will move forward with a small velocity.

These simple equations can be extended to model the impact between different parts of a structural system. These equations also may apply to the closing of gaps between different elastic structures.

3.10 SUMMARY

Several energy methods have been presented that can be used to derive the basic equations used for the static and dynamic analysis of structures. The fundamental equations of structural analysis are equilibrium, force-deformation and compatibility. If the same sign convention is used for element and joint

displacements and forces, the compatibility and equilibrium equations are directly related. If the joint equilibrium equations are written in the same order as the joint forces, the resulting stiffness and flexibility matrices will always be symmetrical.

By assuming displacement shape functions within structural elements, consistent mass and stiffness matrices can be developed. In most cases, however, a physical mass lumping will not produce significant errors.

In dynamic analysis, the independent time integration of the various components of energy, including energy dissipation, can be used to evaluate the accuracy of the solution. By comparing the strain energy stored in the structure resulting from a given load condition, one can modify and improve a structural design to minimize the energy absorbed by the structure

After the structural model has been selected and the loading has been assumed, the structural analysis procedure can be automated. However, the selection of the structural model and the interpretation and verification of the results is the major responsibility of the professional structural engineer.

3.11 REFERENCES

1. Clough, R., and J. Penzien. 1993. *Dynamics of Structures*, Second Edition. McGraw-Hill, Inc. ISBN 0-07-011394-7.