APPENDIX J

CONSISTENT EARTHQUAKE ACCELERATION AND DISPLACEMENT RECORDS

Earthquake Accelerations can be Measured. However, Structures are Subjected to Earthquake Displacements

J.1 INTRODUCTION

{ XE "Acceleration Records" }At the present time most earthquake motions are approximately recorded by accelerometers at equal time intervals. After correcting the acceleration record, as a result of the dynamic properties of the instrument, the record may still contain recording errors. Assuming a linear acceleration within each time interval, a direct integration of the accelerations generally produces a velocity record with a non-zero velocity at the end of the record that should be zero. And an exact integration of the velocity record does not produce a zero displacement at the end of the record. One method currently used to mathematically produce a zero displacement at the end of the record is to introduce a small initial velocity so that the displacement at the end of the record is zero. However, this initial condition is not taken into account in the dynamic analysis of the computer model of the structure. In addition, those displacement records cannot be used directly in multi-support earthquake response analysis.

The purpose of this appendix is to summarize the fundamental equations associated with time history records. It will be demonstrated that the recovery of accelerations from displacements is an unstable numerical operation. A new numerical method is presented for the modification of an acceleration record, or part of an acceleration record, so that it satisfies the fundamental laws of physics in which the displacement, velocity and acceleration records are consistent.

J.2 GROUND ACCELERATION RECORDS

Normally, 200 points per second are used to define an acceleration record, and it is assumed that the acceleration function is linear within each time increment, as shown in Figure J.1.

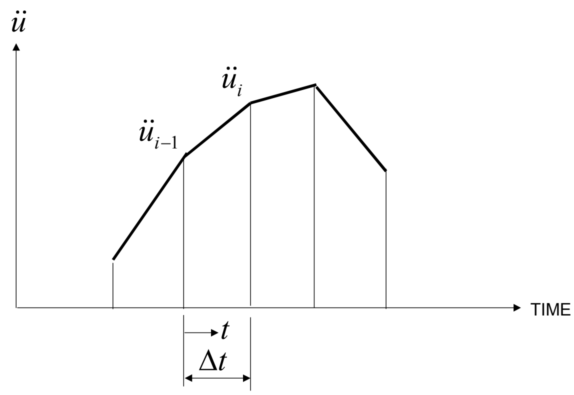

Figure J.1 Typical Earthquake Acceleration Record

Ground velocities and displacements can then be calculated from the integration of the accelerations and velocities within each time step. Or:

$$
\begin{aligned}
\dddot{u} &= \frac{1}{\Delta t}(\ddot{u}_i - \ddot{u}_{i-1}) \\
\ddot{u}(t) &= \ddot{u}_{i-1} + t\,\dddot{u} \\
\dot{u}(t) &= \dot{u}_{i-1} + t\,\ddot{u}_{i-1} + \frac{t^2}{2}\dddot{u} \\
u(t) &= u_{i-1} + t\,\dot{u}_{i-1} + \frac{t^2}{2}\ddot{u}_{i-1} + \frac{t^3}{6}\dddot{u}
\end{aligned}
\tag{J.1}
$$

The evaluation of those equations at $t = \Delta t$ produces the following set of recursive equations:

$$\dddot{u} = \frac{1}{\Delta t}(\ddot{u}_i - \ddot{u}_{i-1})$$
$$\ddot{u}_i = \ddot{u}_{i-1} + \Delta t\,\dddot{u}$$
$$\dot{u}_i = \dot{u}_{i-1} + \Delta t\,\ddot{u}_{i-1} + \frac{\Delta t^2}{2}\dddot{u} \qquad i=1,2,3 \text{ ---------} \qquad \text{(J.2)}$$
$$u_i = u_{i-1} + \Delta t\,\dot{u}_{i-1} + \frac{\Delta t^2}{2}\ddot{u}_{i-1} + \frac{\Delta t^3}{6}\dddot{u}$$

The integration of ground acceleration records should produce zero velocity at the end of the record. In addition, except for near fault earthquake records, zero displacements should be obtained at the end of the record. Real earthquake accelerations are normally corrected to satisfy those requirements.

{ XE "Cubic Displacement Functions" }Note that the displacements are cubic functions within each time increment. Therefore, if displacements are used as the specified seismic loading, smaller time steps or a higher order solution method, based on cubic displacements, must be used for the dynamic structural analysis. On the other hand, if accelerations are used as the basic loading, a lower order solution method, based on linear functions, may be used to solve the dynamic response problem.

J.3 CALCULATION OF ACCELERATION RECORD FROM DISPLACEMENT RECORD

Rewriting Equation (J.2), it should be possible, given the displacement record, to calculate the velocity and acceleration records from the following equations:

$$\dddot{u} = \frac{6}{\Delta t^3}[u_i - u_{i-1} - \Delta t\,\dot{u}_{i-1} - \frac{\Delta t^2}{2}\ddot{u}_{i-1}]$$
$$\dot{u}_i = \dot{u}_{i-1} + \Delta t\,\ddot{u}_{i-1} + \frac{\Delta t^2}{2}\dddot{u} \qquad \text{(J.3)}$$
$$\ddot{u}_i = \ddot{u}_{i-1} + \Delta t\,\dddot{u}$$

On the basis of linear acceleration within each time step, Equations (J.2) and (J.3) are theoretically exact, given the same initial conditions. However, computer round off introduces errors in the velocities and accelerations and the recurrence

Equation (J.3) is unstable and cannot be used to recover the input acceleration record. This instability can be illustrated by rewriting the equations in the following form:

$$\begin{aligned}\dot{u}_i &= -2\,\dot{u}_{i-1} - \frac{\Delta t}{2}\ddot{u}_{i-1} + \frac{3}{\Delta t}(u_i - u_{i-1}) \\ \ddot{u}_i &= -\frac{6}{\Delta t}\dot{u}_{i-1} - 2\,\ddot{u}_{i-1} + \frac{6}{\Delta t^2}(u_i - u_{i-1})\end{aligned} \tag{J.4}$$

If the displacements are constant, the recurrence equation written in matrix form is:

$$\begin{bmatrix}\dot{u}\\ \ddot{u}\end{bmatrix}_i = -\begin{bmatrix}2 & \frac{\Delta t}{2}\\ \frac{6}{\Delta t} & 2\end{bmatrix}\begin{bmatrix}\dot{u}\\ \ddot{u}\end{bmatrix}_{i-1} \tag{J.5}$$

Or, if a small round-off error, ε, is introduced as an initial condition, the following results are produced:

$$\mathbf{u}_0 = \begin{bmatrix}\dot{u}\\ \ddot{u}\end{bmatrix}_0 = \begin{bmatrix}\varepsilon\\ 0\end{bmatrix}, \qquad \mathbf{u}_1 = -\varepsilon\begin{bmatrix}2\\ 6/\Delta t\end{bmatrix}, \qquad \mathbf{u}_2 = \varepsilon\begin{bmatrix}7\\ 24/\Delta t\end{bmatrix} \tag{J.6}$$

It is apparent from Equation (J.6) that the introduction of a small round-off error in the velocity or acceleration at any step will have an opposite sign and be amplified in subsequent time steps. Therefore, it is necessary to use an alternate approach to calculate the velocities and accelerations directly from the displacement record.

{ XE "Spline Functions" }It is possible to use cubic spline functions to fit the displacement data and to recover the velocity and acceleration data. The application of Taylor's series at point *i* produces the following equations for the displacement and velocity:

$$\begin{aligned}u(t) &= u_i + t\,\dot{u}_i + \frac{t^2}{2}\ddot{u}_i + \frac{t^3}{6}\dddot{u} \\ \dot{u}(t) &= \dot{u}_i + t\,\ddot{u}_i + \frac{t^2}{2}\dddot{u}\end{aligned} \tag{J.7}$$

Elimination of $\ddot{u}$ from these equations produces an equation for the acceleration at time t_i. Or:

$$\ddot{u}_i = \frac{6}{t^2}(u_i - u(t)) + \frac{2}{t}(\dot{u}(t) + 2\dot{u}_i) \tag{J.8}$$

Evaluation of Equation (22.10) at $t = \pm\Delta t$ (at $i+1$ and i-1) produces the following equations:

$$\ddot{u}_i = \frac{6}{\Delta t^2}(u_i - u_{i+1}) + \frac{2}{\Delta t}(\dot{u}_{i+1} + 2\dot{u}_i) = \frac{6}{\Delta t^2}(u_i - u_{i-1}) - \frac{2}{\Delta t}(\dot{u}_{i-1} + 2\dot{u}_i) \tag{J.9}$$

Requiring that $\ddot{u}$ be continuous, the following equation must be satisfied at each point:

$$\dot{u}_{i-1} + 4\dot{u}_i + \dot{u}_{i+1} = \frac{3}{\Delta t}(u_{i+1} - u_{i-1}) \tag{J.10}$$

Therefore, there is one unknown velocity per point. This well-conditioned tridiagonal set of equations can be solved directly or by iteration. Those equations are identical to the moment equilibrium equations for a continuous beam that is subjected to normal displacements. After velocities (slopes) are calculated, accelerations (curvatures) and derivatives (shears) are calculated from:

$$\begin{aligned} \ddot{u}_i &= \frac{6}{\Delta t^2}(u_i - u_{i+1}) + \frac{2}{\Delta t}(\dot{u}_i + 2\dot{u}_i) \\ \dddot{u} &= \frac{\ddot{u}_i - \ddot{u}_{i-1}}{\Delta t} \end{aligned} \tag{J.11}$$

This “spline function” approach eliminates the numerical instability problems associated with the direct application of Equations (J.4). However, it is difficult to physically justify how the displacements at a future time point $i+1$ can affect the velocities and accelerations at time point i.

J.4 CREATING CONSISTENT ACCELERATION RECORD

{ XE "Algorithms for:Correction of Acceleration Records" }Earthquake compression, shear and surface waves propagate from a fault rupture at different speeds with the small amplitude compression waves arriving first. For example, acceleration records recorded near the San Francisco-Oakland Bay Bridge from the 1989 Loma Prieta earthquake indicate high frequency, small acceleration motions for the first ten seconds. The large acceleration phase of the record is between 10 and 15 seconds only. However, the official record released covers approximately a 40-second time span. Such a long record is not suitable for a nonlinear, time-history response analysis of a structural model because of the large computer storage and execution time required.

It is possible to select the “large acceleration part of the record” and use it as the basic input for the computer model. To satisfy the fundamental laws of physics, the truncated acceleration record must produce zero velocity and displacement at the beginning and end of the earthquake. This can be accomplished by applying a correction to the truncated acceleration record that is based on the fact that any earthquake acceleration record is a sum of acceleration pluses, as shown in Figure J.2.

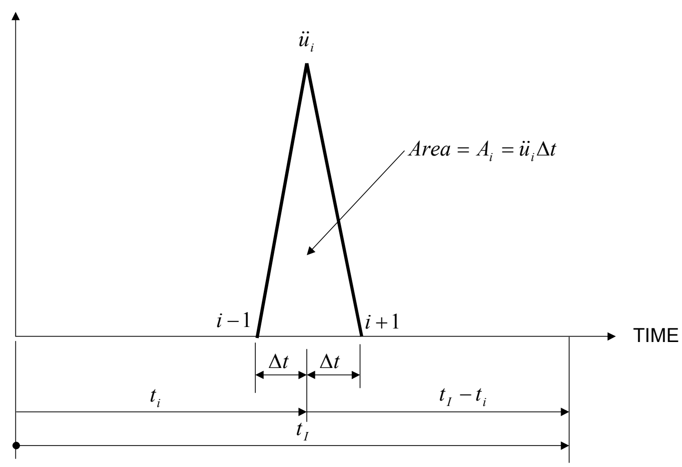

Figure J.2 Typical Earthquake Acceleration Pulse

From spline theory, the exact displacement at the end of the record is given by the following equation:

$$u_I = \sum_{i=1}^{I} (t_I - t_i)\ddot{u}_i \Delta t = \Delta U \tag{J.12}$$

A correction to the acceleration record may now be calculated so that the displacement at the end of the record, Equation (J.12), is identically equal to zero. Rather then apply an initial velocity, the first second or two of the acceleration record can be modified to obtain zero displacement at the end of the record. Let us assume that all of the correction is to be applied to the first "L" values of the acceleration record. To avoid a discontinuity in the acceleration record, the correction will be weighted by a linear function, from α at time zero to zero at time t_L. Therefore, the displacement resulting from the correction function at the end of the record is of the following form:

$$\sum_{i=1}^{L} \alpha \frac{L-i}{L}(t_I - t_i)\ddot{u}_i \Delta t = \alpha_p U_{pos} + \alpha_n U_{neg} = -\Delta U \tag{J.13}$$

For Equation (J.13) the positive and negative terms are calculated separately. If it is assumed that the correction is equal for the positive and negative terms, the amplitudes of the correction constants are given by:

$$\alpha_p = -\frac{2U_{pos}}{\Delta U} \quad \text{and} \quad \alpha_n = -\frac{2U_{neg}}{\Delta U} \qquad \text{(J.14a and J.14b)}$$

Therefore, the correction function can be added to the first "L" values of the acceleration record to obtain zero displacement at the end of the record. This simple correction algorithm is summarized in Table J.1.

If the correction period is less that one second, this very simple algorithm, presented in Table J.1, produces almost identical maximum and minimum displacements and velocities as the mathematical method of selecting an initial velocity. However, this simple one-step method produces physically consistent displacement, velocity and acceleration records. This method does not filter important frequencies from the record and the maximum peak acceleration is maintained.

The velocity at the end of the record can be set to zero if a similar correction is applied to the final few seconds of the acceleration record. Iteration would be required to satisfy both the zero displacement and velocity at the end of the record.

Table J.1 Algorithm to Set Displacement at End of Records to Zero

1. GIVEN UNCORRECTED ACCELERATION RECORD

   $0, \ddot{u}_1, \ddot{u}_2, \ddot{u}_3, \ddot{u}_4, \ldots\ldots\ldots\ldots \ddot{u}_{I-1}, 0 \text{ and L}$

2. COMPUTE CORRECTION FUNCTION

$$\Delta U = \sum_{i=1}^{I} (t_I - t_i)\ddot{u}_i \Delta t$$

$$\sum_{i=0}^{L} \frac{L-i}{L}(t_I - t_i)\ddot{u}_i \Delta t = U_{pos} + U_{neg}$$

$$\alpha_p = -\frac{\Delta U}{2U_{pos}} \quad \text{and} \quad \alpha_n = -\frac{\Delta U}{2U_{neg}}$$

3. CORRECT ACCELERATION RECORD

$$\text{if } \ddot{u}_i > 0 \text{ then } \ddot{u}_i = (1 + \alpha_p \frac{L-i}{L})\ddot{u}_i$$

$$\text{if } \ddot{u}_i < 0 \text{ then } \ddot{u}_i = (1 + \alpha_n \frac{L-i}{L})\ddot{u}_i \qquad i = 1,2....L$$

J.5 SUMMARY

Acceleration records can be accurately defined by 200 points per second and with the assumption that the acceleration is a linear function within each time step. However, the resulting displacements are cubic functions within each time step and smaller time steps must be user-define displacement records. The direct calculation of an acceleration record from a displacement record is a numerically unstable problem, and special numerical procedures must be used to solve this problem.

The mathematical method of using an initial velocity to force the displacement at the end of the record to zero produces an inconsistent displacement record that should not be directly used in a dynamic analysis. A simple algorithm for the correction of the acceleration record has been proposed that produces physically acceptable displacement, velocity and acceleration records.