13.

DYNAMIC ANALYSIS USING MODE SUPERPOSITION

The Mode Shapes used to Uncouple the Dynamic Equilibrium Equations Need Not Be the Exact Free-Vibration Mode Shapes

13.1 EQUATIONS TO BE SOLVED

{ XE "Mode Shapes" }{ XE "Mode Superposition Analysis" }{ XE "Piece-Wise Linear Loading" }The dynamic force equilibrium Equation (12.4) can be rewritten in the following form as a set of N_d second order differential equations:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) = \sum_{j=1}^{J} \mathbf{f}_j \, \mathbf{g}(t)_j \tag{13.1}$$

All possible types of time-dependent loading, including wind, wave and seismic, can be represented by a sum of "J" space vectors $\mathbf{f}_j$, which are not a function of time, and J time functions $\mathbf{g}(t)_j$.

{ XE "Static Condensation" }The number of dynamic degrees of freedom is equal to the number of lumped masses in the system. Many publications advocate the elimination of all massless displacements by static condensation before solution of Equation (13.1). The static condensation method reduces the number of dynamic equilibrium equations to solve; however, it can significantly increase the density and the bandwidth of the condensed stiffness matrix. In building type structures, in which each diaphragm has only three lumped masses, this approach is effective and is automatically used in building analysis programs.

For the dynamic solution of arbitrary structural systems, however, the elimination of the massless displacement is, in general, not numerically efficient. Therefore, the modern versions of the SAP program do not use static condensation to retain the sparseness of the stiffness matrix.

13.2 TRANSFORMATION TO MODAL EQUATIONS

The fundamental mathematical method that is used to solve Equation (13.1) is the separation of variables. This approach assumes the solution can be expressed in the following form:

$$\mathbf{u}(t) = \Phi \mathbf{Y}(t) \tag{13.2a}$$

Where Φ is an "N_d by N" matrix containing N spatial vectors that are not a function of time, and $\mathbf{Y}(t)$ is a vector containing N functions of time.

From Equation (13.2a), it follows that:

$$\dot{\mathbf{u}}(t) = \Phi \dot{\mathbf{Y}}(t) \quad \text{and} \quad \ddot{\mathbf{u}}(t) = \Phi \ddot{\mathbf{Y}}(t) \qquad \text{(13.2b) and (13.2c)}$$

{ XE "Orthogonality Conditions" }Before solution, we require that the space functions satisfy the following mass and stiffness orthogonality conditions:

$$\Phi^T \mathbf{M} \Phi = \mathrm{I} \quad \text{and} \quad \Phi^T \mathbf{K} \Phi = \Omega^2 \tag{13.3}$$

{ XE "Generalized Mass" }{ XE "Mass, Generalized" }where $\mathbf{I}$ is a diagonal unit matrix and Ω^2 is a diagonal matrix in which the diagonal terms are ω_n^2. The term ω_n has the units of radians per second and may or may not be a free vibration frequencies. It should be noted that the fundamentals of mathematics place no restrictions on those vectors, other than the orthogonality properties. In this book each space function vector, ϕ_n, is always normalized so that the ***Generalized Mass*** is equal to one, or $\phi_n^T \mathbf{M} \phi_n = 1.0$.

After substitution of Equations (13.2) into Equation (13.1) and the pre-multiplication by Φ^T, the following matrix of N equations is produced:

$$\mathbf{I}\ddot{\mathbf{Y}}(t) + \mathbf{d}\dot{\mathbf{Y}}(t) + \Omega^2 \mathbf{Y}(t) = \sum_{j=1}^{J} \mathbf{p}_j \, \mathbf{g}(t)_j \qquad (13.4)$$

{ XE "Modal Participation Factors" }where $\mathbf{p}_j = \Phi^T \mathbf{f}_j$ and are defined as the ***modal participation factors*** for load function j. The term p_{nj} is associated with the n^{th} mode. Note that there is one set of "N" modal participation factors for each spatial load condition $\mathbf{f}_j$.

{ XE "Damping Matrix" }{ XE "Damping:Classical Damping" }{ XE "Damping:Modal Damping" }{ XE "Modal Damping" }For all real structures, the "N by N" matrix **d** is not diagonal; however, to uncouple the modal equations, it is necessary to assume ***classical damping*** where there is no coupling between modes. Therefore, the diagonal terms of the modal damping are defined by:

$$d_{nn} = 2\zeta_n \omega_n \qquad (13.5)$$

where ζ_n is defined as the ratio of the damping in mode n to the critical damping of the mode [1].

A typical uncoupled modal equation for linear structural systems is of the following form:

$$\ddot{y}(t)_n + 2\zeta_n \omega_n \dot{y}(t)_n + \omega_n^2 y(t)_n = \sum_{j=1}^{J} p_{nj} \, g(t)_j \qquad (13.6)$$

For three-dimensional seismic motion, this equation can be written as:

$$\ddot{y}(t)_n + 2\zeta_n \omega_n \dot{y}(t)_n + \omega_n^2 y(t)_n = p_{nx}\ddot{u}(t)_{gx} + p_{ny}\ddot{u}(t)_{gy} + p_{nz}\ddot{u}(t)_{gz} \qquad (13.7)$$

{ XE "Earthquake Excitation Factors" }where the three-directional modal participation factors, or in this case ***earthquake excitation factors***, are defined by $p_{nj} = -\phi_n^T \mathbf{M}_j$ in which j is equal to x, y or z and n is the mode number. Note that all mode shapes in this book are normalized so that $\phi_n^T \mathbf{M} \phi_n = 1$.

13.3 RESPONSE DUE TO INITIAL CONDITIONS ONLY

{ XE "Initial Conditions" }Before presenting the solution of Equation (13.6) for various types of loading, it is convenient to define additional constants and functions that are summarized in Table 13.1. This will allow many of the equations presented in other parts of this book to be written in a compact form. Also, the notation reduces the tedium involved in the algebraic derivation and verification of various equations. In addition, it will allow the equations to be in a form that can be easily programmed and verified.

If the "n" subscript is dropped, Equation (13.6) can be written for a typical mode as:

$$\ddot{y}(t) + 2\xi\omega\dot{y}(t) + \omega^2 y(t) = 0 \tag{13.8}$$

in which the initial modal displacement y_0 and velocity $\dot{y}_0$ are specified as a result of previous loading acting on the structure. Note that the functions $S(t)$ and $C(t)$ given in Table 13.1 are solutions to Equation (13.8).

{ XE "Dynamic Response Equations" }**Table 13.1 Summary of Notation used in Dynamic Response Equations**

CONSTANTS

$$\omega_D = \omega\sqrt{1-\xi^2} \qquad \overline{\omega} = \omega\xi \qquad \overline{\xi} = \frac{\xi}{\sqrt{1-\xi^2}}$$

$$a_0 = 2\xi\,\omega \qquad a_1 = \omega_D^2 - \overline{\omega}^2 \qquad a_2 = 2\overline{\omega}\omega_D$$

FUNCTIONS

$$S(t) = e^{-\xi\omega t}\sin(\omega_D t) \qquad C(t) = e^{-\xi\omega t}\cos(\omega_D t)$$

$$\dot{S}(t) = -\overline{\omega}S(t) + \omega_D C(t) \qquad \dot{C}(t) = -\overline{\omega}C(t) - \omega_D S(t)$$

$$\ddot{S}(t) = -a_1 S(t) - a_2 C(t) \qquad \ddot{C}(t) = -a_1 C(t) + a_2 S(t)$$

$$A_1(t) = C(t) + \overline{\xi}S(t) \qquad A_2(t) = \frac{1}{\omega_D}S(t)$$

The solution of Equation (13.8) can now be written in the following compact form:

$$y(t) = A_1(t)y_0 + A_2(t)\dot{y}_0 \tag{13.9}$$

This solution can be easily verified because it satisfies Equation (13.8) and the initial conditions.

13.4 GENERAL SOLUTION DUE TO ARBITRARY LOADING

{ XE "Arbitrary Dynamic Loading" }{ XE "Dynamic Response Equations" }{ XE "Damping:Numerical Damping" }{ XE "Numerical Damping" }{ XE "Time Increment" }There are many different methods available to solve the typical modal equations. However, the use of the exact solution for a load, approximated by a polynomial within a small time increment, has been found to be the most economical and accurate method to numerically solve this equation within computer programs. It does not have problems with stability, and it does not introduce numerical damping. Because most seismic ground accelerations are defined as linear within 0.005 second intervals, the method is exact for this type of loading for all frequencies. Also, if displacements are used as the basic input, the load function derived from linear accelerations are cubic functions within each time interval, as shown in Appendix J.

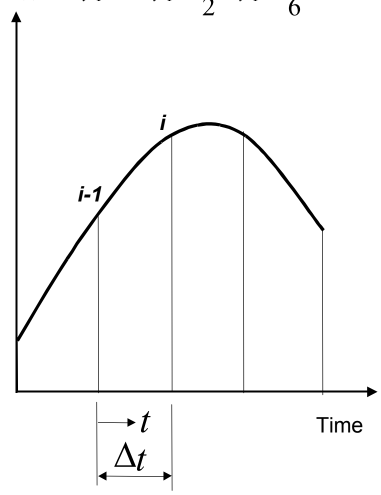
To simplify the notation, all loads are added together to form a typical modal equation of the following form:

$$\ddot{y}(t) + 2\zeta\omega\dot{y}(t) + \omega^2 y(t) = R(t) \tag{13.10}$$

where the modal loading $R(t)$ is a piece-wise polynomial function as shown in Figure 13.1. Note that the higher derivatives required by the cubic load function can be calculated using the numerical method summarized in Appendix J. Therefore, the differential equation to be solved, within the interval $i-1$ to i, is of the following form for both linear and cubic load functions:

$$\ddot{y}(t) + 2\zeta\omega\dot{y}(t) + \omega^2 y(t) = R_{i-1} + t\,\dot{R}_{i-1} + \frac{t^2}{2}\ddot{R}_{i-1} + \frac{t^3}{6}\dddot{R}_{i-1} \tag{13.11}$$

$$R(t) = R_{i-1} + t\,\dot{R}_{i-1} + \frac{t^2}{2}\ddot{R}_{i-1} + \frac{t^3}{6}\dddot{R} \quad \textit{in interval } i\text{-}1 \textit{ to } i$$



For linear loading within interval

$$\dddot{R}_i = 0$$

$$\ddot{R}_i = 0$$

$$\dot{R}_{i-1} = \frac{(R_i - R_{i-1})}{\Delta t}$$

For cubic loading within interval where R_i and $\dot{R}_i$ are specified

$$\ddot{R}_i = \frac{6}{\Delta t^2}(R_i - R_{i+1}) + \frac{2}{\Delta t}(\dot{R}_{i+1} + 2\dot{R}_i)$$

$$\dddot{R} = \frac{\ddot{R}_i - \ddot{R}_{i-1}}{\Delta t}$$

Figure 13.1 Modal Load Functions

From the basic theory of linear differential equations, the general solution of Equation (13.11) is the sum of a homogeneous solution and a particular solution and is of the following form:

$$y(t) = b_1 S(t) + b_2 C(t) + b_3 + b_4 t + b_5 t^2 + b_6 t^3 \qquad (13.12a)$$

The velocity and acceleration associated with this solution are:

$$\dot{y}(t) = b_1 \dot{S}(t) + b_2 \dot{C}(t) + b_4 + 2b_5 t + 3b_6 t^2 \qquad (13.12b)$$

$$\ddot{y}(t) = b_1 \ddot{S}(t) + b_2 \ddot{C}(t) + 2b_5 + 6b_6 t \qquad (13.12c)$$

These equations are summarized in the following matrix equation:

$$\overline{\mathbf{y}}_i = \begin{bmatrix} y_i \\ \dot{y}_i \\ \ddot{y}_i \end{bmatrix} = \begin{bmatrix} S(t) & C(t) & 1.0 & t & t^2 & t^3 \\ \dot{S}(t) & \dot{C}(t) & 0 & 1.0 & 2t & 3t^2 \\ \ddot{S}(t) & \ddot{C}(t) & 0 & 0 & 2.0 & 6t \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = \mathbf{B}(t)\mathbf{b} \tag{13.13}$$

It is now possible to solve for the constants b_i. The initial conditions at $t = 0$ are $\dot{y}(0) = \dot{y}_{i-1}$ and $y(0) = y_{i-1}$. Therefore, from Equations (13.12a and 13.12b)

$$\begin{aligned} \dot{y}_{i-1} &= \omega_D b_1 - \varpi b_2 + b_4 \\ y_{i-1} &= b_2 + b_3 \end{aligned} \tag{13.13a}$$

The substitution of Equations (13.12a, 13.12b and 13.12c) into Equation (13.11) and setting the coefficients of each polynomial term to be equal produce the following four equations:

$$\begin{aligned} 1: &\quad R_{i-1} = \omega^2 b_3 + a_0 b_4 + 2b_5 \\ t: &\quad \dot{R}_{i-1} = \omega^2 b_4 + 2a_0 b_5 + 6b_6 \\ t^2: &\quad \ddot{R}_{i-1} = 2\omega^2 b_5 + 6a_0 b_6 \\ t^3: &\quad \dddot{R}_{i-1} = 6\omega^2 b_6 \end{aligned} \tag{13.13b}$$

These six equations, given by Equations (13.13a and 13.13b), can be written as the following matrix equation:

$$\begin{bmatrix} \dot{y}_{i-1} \\ y_{i-1} \\ R_{i-1} \\ \dot{R}_{i-1} \\ \ddot{R}_{i-1} \\ \dddot{R}_{i-1} \end{bmatrix} = \begin{bmatrix} \omega_D & -\varpi & 0 & 1.0 & 0 & 0 \\ 0 & 1.0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & a_0 & 2.0 & 0 \\ 0 & 0 & 0 & \omega^2 & 2a_0 & 6.0 \\ 0 & 0 & 0 & 0 & 2\omega^2 & 6a_0 \\ 0 & 0 & 0 & 0 & 0 & 6\omega^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} \quad \text{or,} \quad \overline{\mathbf{R}}_{i-1} = \mathbf{C}^{-1}\mathbf{b} \tag{13.14}$$

Therefore,

$$\mathbf{b} = \mathbf{C}\overline{\mathbf{R}}_{i-1} \tag{13.15}$$

The inversion of the upper-triangular matrix **C** can be formed analytically; or it can easily be numerically inverted within the computer program. Hence, the exact solution at time point ***i*** of a modal equation because of a cubic load within the time step is the following:

$$\overline{\mathbf{y}}_i = \mathbf{B}(\Delta t)\mathbf{C}\overline{\mathbf{R}}_{i-1} = \mathbf{A}\overline{\mathbf{R}}_{i-1} \tag{13.16}$$

{ XE "Response Spectrum Analyses:Numerical Evaluation" }{ XE "Duhamel Integral" }{ XE "Time Increment" }Equation (13.16) is a very simple and powerful recursive relationship. The complete algorithm for linear or cubic loading is summarized in Table 13.2. Note that the 3 by 6 **A** matrix is computed only once for each mode. Therefore, for each time increment, approximately 20 multiplications and 16 additions are required. Modern, inexpensive personal computers can complete one multiplication and one addition in approximately 10^{-6} seconds. Hence, the computer time required to solve 200 steps per second for a 50 second duration earthquake is approximately 0.01 seconds. Or 100 modal equations can be solved in one second of computer time. Therefore, there is no need to consider other numerical methods, such as the approximate Fast Fourier Transformation Method or the numerical evaluation of the Duhamel integral, to solve these equations. Because of the speed of this exact piece-wise polynomial technique, it can also be used to develop accurate earthquake response spectra using a very small amount of computer time.

{ XE "Algorithms for:Solution of Modal Equations" }**Table 13.2 Higher-Order Recursive Algorithm for Solution of Modal Equation**

I. EQUATION TO BE SOLVED:

$$\ddot{y}(t) + 2\xi\omega\dot{y}(t) + \omega^2 y(t) = R_{i-1} + t\dot{R}_{i-1} + \frac{t^2}{2}\ddot{R}_{i-1} + \frac{t^3}{6}\dddot{R}_{i-1}$$

II. INITIAL CALCULATIONS

$$\omega_D = \omega\sqrt{1-\xi^2} \qquad \overline{\omega} = \omega\xi \qquad \overline{\xi} = \frac{\xi}{\sqrt{1-\xi^2}}$$

$$a_0 = 2\xi\omega \qquad a_1 = \omega_D^2 - \overline{\omega}^2 \qquad a_2 = 2\overline{\omega}\omega_D$$

$$S(\Delta t) = e^{-\xi\omega\Delta t}\sin(\omega_D \Delta t) \qquad C(\Delta t) = e^{-\xi\omega\Delta t}\cos(\omega_D \Delta t)$$

$$\dot{S}(\Delta t) = -\overline{\omega}S(\Delta t) + \omega_D C(\Delta t) \qquad \dot{C}(\Delta t) = -\overline{\omega}C(\Delta t) - \omega_D S(\Delta t)$$

$$\ddot{S}(\Delta t) = -a_1 S(\Delta t) - a_2 C(\Delta t) \qquad \ddot{C}(\Delta t) = -a_1 C(\Delta t) + a_2 S(\Delta t)$$

$$\mathbf{B}(\Delta t) = \begin{bmatrix} S(\Delta t) & C(\Delta t) & 1.0 & \Delta t & \Delta t^2 & \Delta t^3 \\ \dot{S}(\Delta t) & \dot{C}(\Delta t) & 0 & 1.0 & 2\Delta t & 3\Delta t^2 \\ \ddot{S}(\Delta t) & \ddot{C}(\Delta t) & 0 & 0 & 2.0 & 6\Delta t \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \omega_D & -\overline{\omega} & 0 & 1.0 & 0 & 0 \\ 0 & 1.0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & a_0 & 2.0 & 0 \\ 0 & 0 & 0 & \omega^2 & 2a_0 & 6.0 \\ 0 & 0 & 0 & 0 & 2\omega^2 & 6a_0 \\ 0 & 0 & 0 & 0 & 0 & 6\omega^2 \end{bmatrix}^{-1} \quad \text{and} \quad \mathbf{A} = \mathbf{B}(\Delta t)\mathbf{C}$$

III. RECURSIVE SOLUTION *i=1,2*

a. $\ddot{R}_i = \frac{6}{\Delta t^2}(R_i - R_{i+1}) + \frac{2}{\Delta t}(\dot{R}_{i+1} + 2\dot{R}_i)$

b. $\dddot{R}_{i-1} = \frac{\ddot{R}_i - \ddot{R}_{i-1}}{\Delta t}$

c. $\overline{\mathbf{y}}_i = \mathbf{A}\overline{\mathbf{R}}_{i-1}$

d. *i=i+1* and return to III.a

13.5 SOLUTION FOR PERIODIC LOADING

{ XE "Recurrence Solution for Arbitrary Dynamic Loading" }{ XE "Periodic Dynamic Loading" }{ XE "Wave Loading" }{ XE "Wind Loading" }The recurrence solution algorithm summarized by Equation 13.16 is a very efficient computational method for arbitrary, transient, dynamic loads with initial conditions. It is possible to use this same simple solution method for arbitrary periodic loading as shown in Figure 13.2. Note that the total duration of the loading is from $-\infty$ to $+\infty$ and the loading function has the same amplitude and shape for each typical period T_p. Wind, sea wave and acoustic forces can produce this type of periodic loading. Also, dynamic live loads on bridges may be of periodic form.

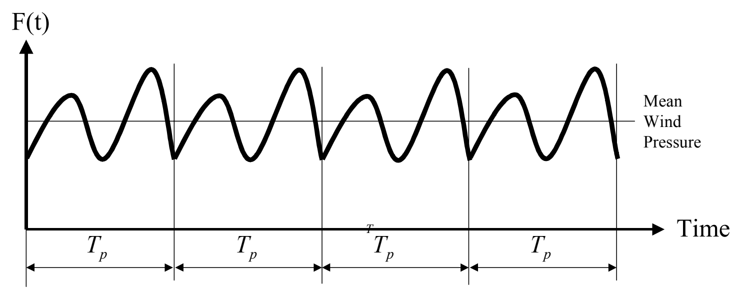


Figure 13.2 Example of Periodic Loading

For a typical duration T_p of loading, a numerical solution for each mode can be evaluated by applying Equation (13.11) without initial conditions. This solution is incorrect because it does not have the correct initial conditions. Therefore, it is necessary for this solution $y(t)$ to be corrected so that the exact solution $z(t)$ has the same displacement and velocity at the beginning and end of each loading period. To satisfy the basic dynamic equilibrium equation, the corrective solution $x(t)$ must have the following form:

$$x(t) = x_0 A_1(t) + \dot{x}_0 A_2(t) \tag{13.17}$$

where the functions are defined in Table 13.1.

The total exact solution for displacement and velocity for each mode can now be written as:

$$z(t) = y(t) + x(t) \tag{13.18a}$$

$$\dot{z}(t) = \dot{y}(t) + \dot{x}(t) \tag{13.18b}$$

So that the exact solution is periodic, the following conditions must be satisfied:

$$z(T_p) = z(0) \tag{13.19a}$$

$$\dot{z}(T_p) = \dot{z}(0) \tag{13.19b}$$

The numerical evaluation of Equation (13.14) produces the following matrix equation, which must be solved for the unknown initial conditions:

$$\begin{bmatrix} 1 - A_1(T_p) & -A_2(T_p) \\ -\dot{A}_1(T_p) & 1 - \dot{A}_2(T_p) \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} -y(T_p) \\ -\dot{y}(T_p) \end{bmatrix} \tag{13.20}$$

The exact periodic solution for modal displacements and velocities can now be calculated from Equations (13.18a and 13.18b). Hence, it in not necessary to use a frequency domain solution approach for periodic loading as suggested in most text books on structural dynamics.

13.6 PARTICIPATING MASS RATIOS

{ XE "Mass Participation Ratios" }{ XE "Participating Mass Ratios" }Several Building Codes require that at least 90 percent of the ***participating mass*** is included in the calculation of response for each principal direction. This requirement is based on a unit base acceleration in a particular direction and calculating the base shear due to that load. The steady state solution for this case involves no damping or elastic forces; therefore, the modal response equations for a unit base acceleration in the x-direction can be written as:

$$\ddot{y}_n = p_{nx} \tag{13.21}$$

The node point inertia forces in the x-direction for that mode are by definition:

$$f_{xn} = \mathrm{M}\ddot{u}(t) = \mathrm{M}\phi_n \ddot{y}_n = p_{nx}\mathrm{M}\phi_n \tag{13.22}$$

The resisting base shear in the x-direction for mode n is the sum of all node point x forces. Or:

$$V_{nx} = -p_{nx}\,\mathrm{I}_x^{\mathrm{T}}\mathrm{M}\phi_n = p_{nx}^2 \tag{13.23}$$

The total base shear in the x-direction, including N modes, will be:

$$V_x = \sum_{n=1}^{N} p_{nx}^2 \tag{13.24}$$

For a unit base acceleration in any direction, the exact base shear must be equal to the sum of all mass components in that direction. Therefore, the ***participating mass ratio*** is defined as the participating mass divided by the total mass in that direction. Or:

$$X_{mass} = \frac{\sum_{n=1}^{N} p_{nx}^2}{\sum m_x} \tag{13.25a}$$

$$Y_{mass} = \frac{\sum_{n=1}^{N} p_{ny}^2}{\sum m_y} \tag{13.25b}$$

$$Z_{mass} = \frac{\sum_{n=1}^{N} p_{nz}^2}{\sum m_z} \tag{13.25c}$$

{ XE "Mass Participation Rule" }If all modes are used, these ratios will all be equal to 1.0. It is clear that the 90 percent participation rule is intended to estimate the accuracy of a solution for base motion only. ***It cannot be used as an error estimator for other types of loading, such as point loads or base displacements acting on the structure***.

Most computer programs produce the contribution of each mode to those ratios. In addition, an examination of those factors gives the engineer an indication of the direction of the base shear associated with each mode. For example, the angle with respect to the x-axis of the base shear associated with the first mode is given by:

$$\theta_1 = \tan^{-1}\left(\frac{p_{1x}}{p_{1y}}\right) \tag{13.26}$$

13.7 STATIC LOAD PARTICIPATION RATIOS

{ XE "Static Load Participation Ratios" }For arbitrary loading, it is useful to determine if the number of vectors used is adequate to approximate the true response of the structural system. One method, which the author has proposed, is to evaluate the static displacements using a truncated set of vectors to solve for the response resulting from static load patterns. As indicated by Equation (13.1), the loads can be written as:

$$\mathbf{F}(t) = \sum_{j=1}^{J} \mathbf{f}_{\mathrm{j}}\, \mathbf{g}(\mathrm{t})_{\mathrm{j}} \tag{13.27}$$

First, one solves the statics problem for the exact displacement $\mathbf{u}_j$ associated with the load pattern $\mathbf{f}_j$. Then, the total external work associated with load condition j is:

$$E_j = \frac{1}{2}\mathbf{f}_j^T \mathbf{u}_j \tag{13.28}$$

From Equation (13.6), the modal response, neglecting inertia and damping forces, is given by:

$$y_n = \frac{1}{\omega_n^2}\phi_n^T \mathbf{f}_j \tag{13.29}$$

From the fundamental definition of the mode superposition method, a truncated set of vectors defines the approximate displacement v_j as:

$$v_j = \sum_{n=1}^{N} y_n \phi_n = \sum_{n=1}^{N} \frac{1}{\omega_n^{\,2}} \phi_n^T \mathbf{f}_j \phi_n \tag{13.30}$$

The total external work associated with the truncated mode shape solution is:

$$\overline{E}_j = \frac{1}{2} \mathbf{f}_j^T \mathbf{v}_j = \sum_{n=1}^{N} \left(\frac{\phi_n^T \mathbf{f}_j}{\omega_n} \right)^2 = \sum_{n=1}^{N} \left(\frac{p_{nj}}{\omega_n} \right)^2 \tag{13.31}$$

A ***static load participation ratio*** r_j can now be defined for load condition j as the ratio of the sum of the work done by the truncated set of modes to the external total work done by the load pattern. Or:

$$r_j = \frac{\overline{E}_j}{E_j} = \frac{\sum_{n=1}^{L} \left(\frac{p_{nj}}{\omega_n} \right)^2}{\mathbf{f}_j^T \mathbf{u}_j} \tag{13.32}$$

If this ratio is close to 1.0, the errors introduced by vector truncation will be very small. However, if this ratio is less than 90 percent, additional vectors should be used in the analysis to capture the ***static load response.***

It has been the experience of the author that the use of exact eigenvectors is not an accurate vector basis for the dynamic analysis of structures subjected to point loads. Whereas, load-dependent vectors, which are defined in the following chapter, always produce a static load participation ratio of 1.0.

13.8 DYNAMIC LOAD PARTICIPATION RATIOS

{ XE "Dynamic Participation Ratios" }In addition to participating mass ratios and static load participation ratios, it is possible to calculate a ***dynamic load participation ratio*** for each load pattern. All three of these ratios are automatically produced by the SAP2000 program.

The dynamic load participation ratio is based on the physical assumption that only inertia forces resist the load pattern. Considering only mass degrees of freedom, the exact acceleration $\ddot{\mathbf{u}}_j$ because of the load pattern $\mathbf{f}_j$ is:

$$\ddot{\mathbf{u}}_j = \mathbf{M}^{-1}\mathbf{f}_j \tag{13.33}$$

The velocity of the mass points at time $t = 1$ is:

$$\dot{\mathbf{u}}_j = t\mathbf{M}^{-1}\mathbf{f}_j = \mathbf{M}^{-1}\mathbf{f}_j \tag{13.34}$$

Hence, the total kinetic energy associated with load pattern j is:

$$E_j = \frac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}} = \frac{1}{2}\mathbf{f}_j^T\mathbf{M}^{-1}\mathbf{f}_j \tag{13.35}$$

From Equation 13.6, the modal acceleration and velocity, neglecting the massless degrees of freedom, is given by:

$$\ddot{y}_n = \phi_n^T\mathbf{f}_j \quad \text{and}\; \dot{y}_n = t\phi_n^T\mathbf{f}_j = \phi_n^T\mathbf{f}_j \;\; \text{at} \;\; t = 1 \tag{13.36}$$

From the fundamental definition of the mode superposition method, a truncated set of vectors defines the approximate velocity $\dot{\mathbf{v}}_j$ as:

$$\dot{\mathbf{v}}_j = \sum_{n=1}^{N}\dot{y}_n\phi_n = \sum_{n=1}^{N}\phi_n^T\mathbf{f}_j\phi_n = \sum_{n=1}^{N}p_{nj}\,\phi_n = \sum_{n=1}^{N}\phi_n\,p_{nj} \tag{13.37}$$

The total kinetic energy associated with the truncated mode shape solution is:

$$\overline{E}_j = \frac{1}{2}\dot{\mathbf{v}}_j^T\mathbf{M}\dot{\mathbf{v}}_j = \frac{1}{2}\sum_{n=1}^{N}p_{nj}\phi_n{}^T\;\mathbf{M}\sum_{n=1}^{N}\phi_n\,p_{nj} = \frac{1}{2}\sum_{n=1}^{N}(p_{nj})^2 \tag{13.38}$$

A ***dynamic load participation ratio*** r_j can now be defined for load condition j as the ratio of the sum of the kinetic energy associated with the truncated set of modes to the total kinetic energy associated with the load pattern. Or:

$$r_j = \frac{\overline{E}_j}{E_j} = \frac{\sum_{n=1}^{N}(p_{nj})^2}{\mathbf{f}_j^T\mathbf{M}^{-1}\mathbf{f}_j} \tag{13.39}$$

The dynamic load participation ratio includes only loads that are associated with mass degrees of freedom. However, the static load participation factor includes the effects of the loads acting at the massless degrees of freedom.

A 100 percent dynamic load participation indicates that the high frequency response of the structure is captured. In addition, for the cases of mass proportional loading in the three global directions, the dynamic load participation ratios are identical to the mass participation factors.

13.9 SUMMARY

The mode superposition method is a very powerful method used to reduce the number of unknowns in a dynamic response analysis. All types of loading can be accurately approximated by piece-wise linear or cubic functions within a small time increment. Exact solutions exist for these types of loading and can be computed with a trivial amount of computer time for equal time increments. Therefore, there is no need to present other methods for the numerical evaluation of modal equations.

To solve for the linear dynamic response of structures subjected to periodic loading, it is only necessary to add a corrective solution to the transient solution for a typical time period of loading. The corrective solution forces the initial conditions of a typical time period to be equal to the final conditions at the end of the time period. Hence, the same time-domain solution method can be used to solve wind or wave dynamic response problems in structural engineering.

Participating mass factors can be used to estimate the number of vectors required in an elastic seismic analysis where base accelerations are used as the fundamental loading. The use of mass participation factors to estimate the accuracy of a nonlinear seismic analysis can introduce significant errors. Internal nonlinear concentrated forces that are in equal and opposite directions do not produce a base shear. In addition, for the case of specified base displacements, the participating mass ratios do not have a physical meaning.

Static and dynamic participation ratios are defined and can be used to estimate the number of vectors required. It will later be shown that the use of Ritz vectors, rather than the exact eigenvectors, will produce vectors that have static and dynamic participation ratios at or near 100 percent.